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Weighted-Set Graph Colorings

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Abstract We study a weighted-set graph coloring problem in which one assigns q colors to the vertices of a graph such that adjacent vertices have different colors, with a vertex weighting w that either disfavors or favors a given subset of s colors contained in the set of q colors. We construct and analyze a weighted-set chromatic polynomial $Ph(G, q, s, w)$ associated with this coloring. General properties of this weighted-set chromatic polynomial are proved, and illustrative calculations are presented for various families of graphs. This study extends a previous one for the case $s = 1$ and reveals a number of interesting new features.

Keywords Potts model · Graph coloring

1 Introduction

Recently, two weighted graph coloring problems have been formulated and studied in which one assigns q colors to the vertices of a graph subject to the condition that adjacent vertices (i.e., vertices connected by an edge of the graph) have different colors, with a vertex weighting w that either disfavors (for $0 \le w < 1$) or favors (for $w > 1$) a given color [1, 2]. Since all of the colors are, *a priori*, equivalent, it does not matter which color one takes to be given the weighting. An assignment of q colors to the vertices of a graph G, such that adjacent vertices have different colors, is called a "proper q-coloring" of the vertices of G. In the present paper we shall study a generalization of this problem in which one performs a proper q-coloring of the vertices of a graph G such that s colors are favored or disfavored relative to the remaining $q - s$ colors. We denote these coloring problems as the DFSCP and FSCP for

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disfavored or favored weighted-set graph vertex coloring problems. We analyze the properties of an associated weighted-set chromatic polynomial, denoted $Ph(G, q, s, w)$, which generalizes the chromatic polynomial $P(G, q)$ and the single-color weighted chromatic polynomial $Ph(G, q, w) \equiv Ph(G, q, 1, w)$ analyzed in Ref. [2]. We shall denote the set of integers $\{1, \dots, s\}$, representing colors, as I_s and the orthogonal complement $\{s+1, \dots, q\}$ as $I_s^{\perp}$. To each proper q-coloring of the vertices of a graph G there corresponds a term w^{n_s}, where n_s denotes the number of vertices assigned a color in I_s. The sum of such terms resulting from all of these proper q-colorings of the vertices of G is the function $Ph(G, q, s, w)$. As we shall show below, this is a polynomial not only in w, but also in q and s. This polynomial constitutes a w-dependent measure, extended from the integers to the real numbers, of the number of proper q-colorings of the vertices of G. In the weighted-set graph coloring problem for a given graph G, with $q \in \mathbb{N}_+$ being the number of colors, $Ph(G, q, s, w)$ is a map from $(q, s, w) \in \mathbb{N}_+ \times I_s \times [0, \infty)$ to $\mathbb{R}$. One can formally extend the domain of each of the variables q, s, and w to $\mathbb{R}$ or, indeed, $\mathbb{C}$, and the latter extension is necessary when one analyzes the zeros of $Ph(G, q, s, w)$. The polynomial $Ph(G, q, s, w)$ is equivalent to the partition function of the q-state Potts antiferromagnet on the graph G in a set of external magnetic fields, in the limit where the effective spin-spin exchange coupling becomes infinitely strong, so that the only spin configurations contributing to this partition function are those for which spins on adjacent vertices are different [1–3]. There has been continuing interest in the Potts model and chromatic and Tutte polynomials for many years; reviews of the Potts model include [4–7] and reviews of chromatic and Tutte polynomials include [8–16].

There are several motivations for this study, arising from the areas of mathematics, physics, and engineering. One motivation is the intrinsic mathematical interest in graph coloring problems and the fact that there seems to have been very little previous study of weighted-set graph coloring. A second one stems from the equivalence to the statistical mechanics of the Potts antiferromagnet in a set of magnetic fields that disfavor or favor a corresponding set of spin values. A third reason for interest in this subject is the fact that these weighted-set graph coloring problems have practical applications. For example, the weighted graph coloring problem with $0 \le w < 1$ (i.e., the DFSCP) describes, among other things, the assignment of frequencies to commercial radio broadcasting stations in an area such that (i) adjacent stations must use different frequencies to avoid interference and (ii) stations prefer to avoid transmitting on a set of s specific frequencies, e.g., because these are used for data-taking by a nearby radio astronomy antenna. The weighted graph coloring problem with $w > 1$ (i.e., the FSCP) describes this frequency assignment process with a preference for a set of s frequencies, e.g., because these are most free of interference. We shall especially emphasize the connections with the first two of these areas in this paper.

We note some special cases of the weighted-set chromatic polynomial. Let us consider a graph $G = (V, E)$, defined by its set of vertices V and edges (= bonds) E. We denote the numbers of vertices and edges of G as $n(G) \equiv n$ and $e(G)$. The values $s = 0$ and $w = 1$ correspond to the usual unweighted proper q-coloring of the vertices of G, so $Ph(G, q, s, w)$ reduces to the usual chromatic polynomial counting the number of proper q-colorings of the vertices of G:

$$Ph(G, q, s, 1) = Ph(G, q, 0, w) = P(G, q). \tag{1.1}$$

Since the right-hand side of (1.1) is independent of s and w, this relation also implies the differential equations

$$\frac{\partial Ph(G, q, s, 1)}{\partial s} = 0 \tag{1.2}$$

and

$$\frac{\partial Ph(G,q,0,w)}{\partial w} = 0. \tag{1.3}$$

For $w = 0$, one is prevented from assigning any of the s disfavored colors to any of the vertices, so that the problem reduces to that of a proper coloring of the vertices of G with $q - s$ colors, without any weighting among them. This is described by the usual (unweighted) chromatic polynomial $P(G, q - s)$, so

$$Ph(G,q,s,0) = P(G,q-s). \tag{1.4}$$

Thus, the DFSCP, described by $Ph(G,q,s,w)$ may be regarded as interpolating between $P(G,q)$ and $P(G,q-s)$ as w decreases through real values from $w = 1$ to $w = 0$. (The case of no weighting, $w = 1$, may be considered to be the border between the DFSCP and FSCP regimes.) If $s = q$, so that all of the colors receive the same weighting, then, as is clear from its definition, the weighted-set chromatic polynomial reduces to w^n times the unweighted chromatic polynomial:

$$Ph(G,q,q,w) = w^n P(G,q). \tag{1.5}$$

Thus, for $s = 0$ and $s = q$, $Ph(G,q,s,w)$ reduces to 1 and w^n times $P(G,q)$, respectively, while for other values of s, in particular, for integer s in the interval $1 \le s \le q-1$, $Ph(G,q,s,w)$ is a new polynomial which is not, in general, reducible to $P(G,q)$. Hence, while retaining the term DFSCP, we shall often focus on the new cases where w lies strictly between 1 and 0. As we shall show below, the weighted-set chromatic polynomial $Ph(G,q,s,w)$ satisfies a basic symmetry relation involving the interchange of s with $q-s$, so that a knowledge of the weighted-set proper q-coloring of the vertices of a graph G with a set of s colors is equivalent to a knowledge of the proper coloring of the vertices of G with a set of $q-s$ colors.

There are important differences between the case $s = 1$ studied previously in Ref. [2] and the cases $2 \le s \le q$. For $s = 1$, as w increases above 1 to large positive values, the favored weighting of one color is increasingly in conflict with the strict constraint that no two adjacent vertices have the same color. Hence, this involves competing interactions and frustration. In contrast, in the FSCP regime with $s \ge 2$, depending on the graph G, one may avoid this conflict and the resultant frustration. Specific differences will be apparent in our explicit results. For example, in our general result for $Z(G,q,s,v,w)$ for the circuit graph C_n in (5.5) below, one term vanishes identically in the case $s = 1$ but is present for other values of s in the interval I_s. (By the $s \leftrightarrow q-s$ symmetry in (2.4) and (2.13), this also means that another term vanishes identically for $s = q-1$ but is present for other values of $s \in I_s$.)

2 Some Basic Properties

2.1 Connection of $Ph(G,q,s,w)$ with Statistical Mechanics

It is useful to see how the function $Ph(G,q,s,w)$ arises in a more general statistical mechanical context. As before, we have $G = (V,E)$. A spanning subgraph $G' \subseteq G$ is defined as the subgraph containing the same set of vertices V and a subset of the edges of G; $G' = (V,E')$ with $E' \subseteq E$. We denote the number of connected components of G as $k(G)$

and the connected subgraphs of a spanning subgraph G' as G'_i, $i = 1, \dots, k(G')$. To obtain an expression for $Ph(G, q, s, w)$, we make use of the fact that it is a special case of the partition function for the q-state Potts model in the presence of external magnetic fields in the limit of infinitely strong antiferromagnetic spin-spin coupling. In thermal equilibrium at temperature T, the general Potts model partition function is given by

$$Z = \sum_{\{\sigma_n\}} e^{-\beta \mathcal{H}} \tag{2.1}$$

with the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \delta_{\sigma_i, \sigma_j} - \sum_{p=1}^{q} H_p \sum_{\ell} \delta_{\sigma_\ell, p}, \tag{2.2}$$

where i, j, ℓ label vertices of G, σ_i are classical spin variables on these vertices, taking values in the set $I_q = \{1, \dots, q\}$, $\beta = (k_B T)^{-1}$, $\langle ij \rangle$ denote pairs of adjacent vertices, p is an integer, $p \in I_q$, and H_p is an external magnetic field given by

$$H_p = \begin{cases} H \neq 0 & \text{for } 1 \le p \le s \\ 0 & \text{for } s + 1 \le p \le q. \end{cases} \tag{2.3}$$

The zero-field Potts model Hamiltonian $\mathcal{H}$ and partition function Z are invariant under the global transformation in which $\sigma_i \to g\sigma_i$ $\forall i \in V$, with $g \in S_q$, where S_q is the symmetric (= permutation) group on q objects. Because of this invariance, we can, without loss of generality, take the external magnetic fields H_p to single out a set of s contiguous spin values (equivalently, colors) $\sigma_i \in I_s$ as disfavored or favored, relative to the orthogonal complement of values $\sigma_i \in I_s^\perp$. In the presence of the magnetic fields H_p given in (2.3), the symmetry group of $\mathcal{H}$ and Z is reduced to the tensor product

$$S_q \to S_s \otimes S_{q-s}. \tag{2.4}$$

That is, if $g_1 \in S_s$ and $g_2 \in S_{q-s}$, then the global transformation $\sigma_i \to (g_1 \otimes g_2)\sigma_i$ $\forall i$ leaves $\mathcal{H}$ and Z invariant. Here $(g_1 \otimes g_2)\sigma_i$ means $g_1\sigma_i$ if $\sigma_i \in I_s$ and $g_2\sigma_i$ if $\sigma_i \in I_s^\perp$.

Let us introduce the notation

$$K = \beta J, \qquad h = \beta H, \qquad y = e^K, \qquad v = y - 1, \qquad w = e^h. \tag{2.5}$$

Thus, the physical ranges of v are $v \ge 0$ for the Potts ferromagnet, and $-1 \le v \le 0$ for the Potts antiferromagnet. The weighted-set chromatic polynomial is then obtained by choosing the antiferromagnetic sign of the spin-spin coupling, $J < 0$ and taking $K \to -\infty$ while keeping $h = \beta H$ fixed. Since $K = \beta J$, the limit $K \to -\infty$ results if (i) one takes $J \to -\infty$ while holding T and H fixed and finite, or (ii) one takes $T \to 0$, i.e., $\beta \to \infty$, with J fixed and finite and $H \to 0$ so as to keep $h = \beta H$ fixed and finite. The limit $K \to -\infty$ guarantees that no two adjacent spins have the same value, or, in the coloring context, that no two adjacent vertices have the same color. One sees that in this statistical mechanics context, it is the external magnetic fields that produce the weighting that favors or disfavors a given set I_s of spin values. Positive H gives a weighting that favors spin configurations in which spins have values in the set I_s, or equivalently, vertex colorings with colors in this set, while negative H disfavors such configurations. For positive and negative H, the corresponding ranges of w are $w > 1$ and $0 \le w < 1$, respectively.

In Ref. [2] a formula was derived for the partition function Z which does not make any explicit reference to the spins σ_i or the summation over spin configurations, but instead expresses this function as a sum of terms arising from the $2^{e(G)}$ spanning subgraphs $G' \subseteq G$, namely

$$Z(G, q, s, v, w) = \sum_{G' \subseteq G} v^{e(G')} \prod_{i=1}^{k(G')} \left(q - s + s w^{n(G'_i)}\right). \tag{2.6}$$

This generalizes a spanning subgraph formula for Z in the case $s = 1$ due to F.Y. Wu [17], which, itself, generalized the Fortuin-Kasteleyn formula for the zero-field Potts model partition function [18],

$$Z(G, q, v) = \sum_{G' \subseteq G} v^{e(G')} q^{k(G')}. \tag{2.7}$$

The original definition of the Potts model, (2.1) and (2.2), requires q to be in the set of positive integers $\mathbb{N}_+$ and s to be a non-negative integer. These restrictions are removed by (2.6). Furthermore, (2.6) shows that Z is a polynomial in the variables q, s, v, and w, hence our notation $Z(G, q, s, v, w)$.

The $K \to -\infty$ limit that yields the weighted-set chromatic polynomial is equivalent to $v = -1$, so

$$Ph(G, q, s, w) = Z(G, q, s, -1, w). \tag{2.8}$$

Hence,

$$Ph(G, q, s, w) = \sum_{G' \subseteq G} (-1)^{e(G')} \prod_{i=1}^{k(G')} \left(q - s + s w^{n(G'_i)}\right). \tag{2.9}$$

We recall the factorization

$$w^m - 1 = (w - 1) \sum_{j=0}^{m-1} w^j \tag{2.10}$$

and apply it to (2.6) with $m = n(G'_i)$. Since the variable s only appears in (2.6) in the form

$$\prod_{i=1}^{k(G')} \left(q - s + s w^{n(G'_i)}\right) = \prod_{i=1}^{k(G')} \left(q + s(w - 1) \sum_{r=0}^{n(G'_i)-1} w^r\right), \tag{2.11}$$

it follows that $Z(G, q, s, v, w)$ and $Ph(G, q, s, w)$ can equivalently be written as polynomials in the variables q, v, w, and

$$t = s(w - 1). \tag{2.12}$$

The advantage of doing this is that it shortens expressions for these polynomials; however, it renders the symmetries (2.13) and (2.14) below not manifest in the resultant expressions.

Having shown the connection of $Ph(G, q, s, w)$ to $Z(G, q, s, v, w)$, we observe that various properties of $Ph(G, q, s, w)$ can be expressed more generally as corresponding properties of $Z(G, q, s, v, w)$. From (2.6) it follows that the Potts model partition function $Z(G, q, s, v, w)$ satisfies a basic symmetry relating the values s and $q - s$:

$$Z(G, q, s, v, w) = w^n Z(G, q, q - s, v, w^{-1}) \tag{2.13}$$

so that, in particular, setting $v=-1$,

$$Ph(G,q,s,w)=w^n Ph(G,q,q-s,w^{-1}). \tag{2.14}$$

The symmetry relation (2.13) is obvious from a statistical mechanics context as well as from the formula (2.6); it is a statement of the fact that the presence of the magnetic field disfavors or favors the set of spin values $\sigma_i \in I_s$ relative to the orthogonal complement of spin values $\sigma_i \in I_s^{\perp}$, but, up to the prefactor, this is equivalent to replacing s by $q-s$ and reversing the sign of H, i.e., replacing w by $1/w$.

If the magnetic field is zero, i.e., $w=1$, or $s=0$, so that no spin values are weighted differently by this field, we have

$$Z(G,q,s,v,1)=Z(G,q,0,v,w)=Z(G,q,v), \tag{2.15}$$

so that

$$\frac{\partial Z(G,q,s,v,1)}{\partial s}=0 \tag{2.16}$$

and

$$\frac{\partial Z(G,q,0,v,w)}{\partial w}=0. \tag{2.17}$$

If $s=q$, so that all spin values receive the same weighting, then

$$Z(G,q,q,v,w)=w^n Z(G,q,v). \tag{2.18}$$

Note that this result also follows by applying the symmetry relation (2.13), so that $Z(G,q,q,v,w)=w^n Z(G,q,0,v,w^{-1})=w^n Z(G,q,v)$. Moreover, if the disfavoring is total, i.e., $w=0$, then

$$Z(G,q,s,v,0)=Z(G,q-s,v). \tag{2.19}$$

From the definition of $Ph(G,q,s,w)$ as a sum of terms w^{n_s} corresponding to proper q-colorings of the vertices of the graph G such that n_s vertices are assigned colors in the weighted set I_s, we can infer a general inequality. Let us denote the total set of proper q-colorings of the vertices of G as $\{\sigma\}$ and a subset as $\{\sigma\}_{subset}$, and let us define $Ph(G,q,s,w)_{subset}$ as the sum of terms w^{n_s} resulting from the contributions of the subset $\{\sigma\}_{subset}$ of proper q-colorings of the vertices of G. Then, since for the coloring problem at hand, where $w \ge 0$, each such proper q-coloring contributes a non-negative term to $Ph(G,q,s,w)$, we have the general inequality

$$Ph(G,q,s,w) \ge Ph(G,q,s,w)_{subset}. \tag{2.20}$$

2.2 Some Properties Connected with Characteristics of Graphs

If G has any loop, defined as an edge that connects a vertex to itself, then a proper q-coloring is impossible. This is because such a q-coloring requires that any two adjacent vertices have different colors, but since the vertices connected by an edge are adjacent, the presence of a loop in G means that a vertex is adjacent to itself. Thus, $Ph(G,q,s,w)=0$ if G contains a loop. Hence, with no loss of generality, in our discussions of $Ph(G,q,s,w)$ we shall restrict our analysis in this paper to loopless graphs G. Thus, in the text below, where $G=(V,E)$

is characterized as having a non-empty edge set $E \neq \emptyset$, it is understood that E does not contain any loops.

Another basic property of a chromatic polynomial is that as long as two vertices are joined by an edge, adding more edges connecting these vertices does not change the chromatic polynomial. This is clear from the fact that the chromatic polynomial counts the number of proper q-colorings of the vertices of G, and the relevant condition—that two adjacent vertices must have different colors—is the same regardless of whether one or more than one edges join these vertices. Let us define an operation of "reduction of multiple edge(s)" in G, denoted $R_E(G)$, as follows: if two vertices are joined by a multiple edge, then delete all but one of these edges, and carry out this reduction on all edges, so that the resultant graph $R_E(G)$ has only single edges. Then if G is a graph that contains one or more multiple edges joining some set(s) of vertices, $P(G,q) = P(R_E(G),q)$. Since the same proper q-coloring condition holds for the weighted-set chromatic polynomial, we have

$$Ph(G,q,s,w) = Ph(R_E(G),q,s,w). \tag{2.21}$$

Moreover, if G consists of two disjoint parts, G_1 and G_2, then $Z(G,q,s,v,w)$ is simply the product $Z(G,q,s,v,w) = Z(G_1,q,s,v,w)Z(G_2,q,s,v,w)$, and the same factorization property holds for the special case $v=-1$ that yields $Ph(G,q,s,w)$. Hence, without loss of generality, unless otherwise indicated, we shall restrict our discussion here to connected graphs G.

2.3 Properties of Coefficients in Polynomial Expansions

Next, we prove some general structural properties of $Z(G,q,s,v,w)$ and $Ph(G,q,s,w)$ that hold for an arbitrary graph G. From (2.6) and (2.9) one can derive certain factorization properties of these polynomials. It is convenient to define the notation

$$\tilde{q} = q - s. \tag{2.22}$$

From (2.6), it follows that we can write $Z(G,q,s,v,w)$ in several equivalent ways:

$$\begin{aligned} Z(G,q,s,v,w) &= \sum_{i,j,\ell=0}^{n} \sum_{k=0}^{e(G)} a_{i,j,k,\ell} q^i s^j v^k w^\ell = \sum_{i,j,\ell=0}^{n} \sum_{k=0}^{e(G)} b_{i,j,k,\ell} q^i s^j y^k w^\ell \\ &= \sum_{i,j,\ell=0}^{n} \sum_{k=0}^{e(G)} c_{i,j,k,\ell} \tilde{q}^i s^j v^k w^\ell = \sum_{i,j,\ell=0}^{n} \sum_{k=0}^{e(G)} d_{i,j,k,\ell} q^i t^j v^k w^\ell, \end{aligned} \tag{2.23}$$

where $a_{i,j,k,\ell}$, $b_{i,j,k,\ell}$, $c_{i,j,k,\ell}$, and $d_{i,j,k,\ell}$ are integers (and i, j, k, ℓ are dummy summation variables here). Some $a_{i,j,k,\ell}$ and $b_{i,j,k,\ell}$ can be negative, but the nonzero $c_{i,j,k,\ell}$ and $d_{i,j,k,\ell}$ are positive, as follows from (2.6) and (2.11). From these equations, one infers corresponding ones for $Ph(G,q,s,w)$ by setting $v=-1$, i.e., $y=0$.

For our analysis below and for comparisons with chromatic polynomials, three types of polynomial expansions will be useful. Because of the basic symmetry (2.13), the most useful expansion of $Z(G,q,s,v,w)$ is as a sum of powers of w with coefficients, denoted as $\beta_{G,j}(q,s,v)$, which are polynomials in q, s, and v:

$$Z(G,q,s,v,w) = \sum_{j=0}^{n} \beta_{Z,G,j}(q,s,v) w^j. \tag{2.24}$$

The symmetry (2.13) implies the following relation among the coefficients:

$$\beta_{Z,G,j}(q,s,v) = \beta_{Z,G,n-j}(q,q-s,v) \quad \text{for } 0 \le j \le n. \tag{2.25}$$

In particular, for the special case $v=-1$ of primary interest here, we write

$$Ph(G,q,s,w) = \sum_{j=0}^{n} \beta_{G,j}(q,s) w^j, \tag{2.26}$$

where

$$\beta_{G,j}(q,s) \equiv \beta_{Z,G,j}(q,s,-1). \tag{2.27}$$

From (2.25), we have

$$\beta_{G,j}(q,s) = \beta_{G,n-j}(q,q-s) \quad \text{for } 0 \le j \le n. \tag{2.28}$$

If n is even, say $n=2m$, then the middle coefficient is transformed into itself, giving rise to the results that

$$\text{If } n=2m \text{ is even, then} \quad \begin{aligned} \beta_{Z,G,m}(q,s,v) &= \beta_{Z,G,m}(q,q-s,v) \\ \beta_{G,m}(q,s) &= \beta_{G,m}(q,q-s). \end{aligned} \tag{2.29}$$

From (2.6), it is clear that the term of highest degree in w arises from products of the $sw^{n(G'_i)}$ factors over the various connected components G'_i for each spanning subgraph $G' \subseteq G$, and then over the spanning subgraphs G'. This product does not involve q, so that

$$\beta_{Z,G,n}(q,s,v) \text{ and } \beta_{G,n}(q,s) \text{ are independent of } q. \tag{2.30}$$

The $\beta_{Z,G,j}(q,s,v)$ coefficients have especially simple factorization properties, which we analyze next. Evaluating (2.24) at $w=0$, where only the w^0 term remains, and combining this evaluation with the relation (2.19), we derive the result

$$\beta_{Z,G,0}(q,s,v) = Z(G,q-s,v). \tag{2.31}$$

Combining the relation for $w=1$ in (2.15) with (2.24), we derive a formula for the sum of the coefficients $\beta_{Z,G,j}(q,s,v)$:

$$\sum_{j=0}^{n} \beta_{Z,G,j}(q,s,v) = Z(G,q,v). \tag{2.32}$$

Because this sum is independent of s, (2.32) also yields the differential equation

$$\frac{\partial}{\partial s} \sum_{j=0}^{n} \beta_{Z,G,j}(q,s,v) = 0. \tag{2.33}$$

Next, we set $s=q$ in (2.24) and use (2.18). Since the resulting expression must be proportional to w^n, all of the coefficients of the terms in $Z(G,q,s,v,w)$ of lower degree in w than n must vanish. Because these coefficients $\beta_{Z,G,j}(q,s,v)$ are polynomials in q and s (as well as v), this means that they must contain the factor $(q-s)$:

$$\beta_{Z,G,j}(q,s,v) \text{ and } \beta_{G,j}(q,s) \text{ contain the factor } (q-s) \text{ for } 0 \le j \le n-1. \tag{2.34}$$

Furthermore, using (2.18) for this $s = q$ evaluation, we infer that $\beta_{Z,G,n}(q,q,v) = Z(G,q,v)$. However, since, by (2.30), $\beta_{Z,G,n}(q,s,v)$ is independent of q and is only a function of s and v, this implies that

$$\beta_{Z,G,n}(q,s,v) = Z(G,s,v) \tag{2.35}$$

(so we could drop the argument q, but for uniformity with other coefficients $\beta_{Z,G,j}(q,s,v)$, we shall retain it).

Now setting $s = 0$ reduces $Z(G,q,s,v,w)$ to $Z(G,q,v)$ (cf. (2.15)). Since the w^0 term (given in (2.31)) is, by itself, equal to $Z(G,q,v)$ for $s = 0$, this means that all of the other terms proportional to nonzero powers w^j, $j = 1,\dots,n$ in $Z(G,q,s,v,w)$ must vanish when $s = 0$. This proves that

$$\text{For } 1 \le j \le n,\ \beta_{Z,G,j}(q,s,v) \text{ and } \beta_{G,j}(q,s) \text{ contain a factor of } s. \tag{2.36}$$

Various special cases of these results for the weighted-set chromatic polynomial are obtained by setting $v = -1$ in the requisite equations. Thus, (2.31) implies

$$\beta_{G,0}(q,s) = P(G,q-s), \tag{2.37}$$

and (2.35) implies

$$\beta_{G,n}(q,s) = P(G,s). \tag{2.38}$$

We now focus on $Ph(G,q,s,w)$. The chromatic number of G, denoted $\chi(G)$, is the minimal number of colors for which one can carry out a proper q-coloring of the vertices of G. Since the proper q-coloring constraint cannot be satisfied for integer q in the interval $0 \le q \le \chi(G) - 1$, the chromatic polynomial $P(G,q)$ vanishes for these values and hence contains $\prod_{j=0}^{\chi(G)-1}(q-j)$ as a factor. Applying this to (2.38) shows that

$$\beta_{G,n}(q,s) \text{ contains the factor } \prod_{j=0}^{\chi(G)-1}(s-j), \tag{2.39}$$

and applying it to (2.37), taking into account the shift $s \to q - s$, shows that

$$\beta_{G,0}(q,s) \text{ contains the factor } \prod_{j=0}^{\chi(G)-1}(q-s-j). \tag{2.40}$$

In particular, provided that $G = (V,E)$ contains at least one edge, so that $\chi(G) \ge 2$, we have the results

$$\text{If } E \ne \emptyset, \text{ then } \beta_{G,n}(q,s) \text{ contains a factor } s(s-1) \tag{2.41}$$

and

$$\text{If } E \ne \emptyset, \text{ then } \beta_{G,0}(q,s) \text{ contains a factor } (q-s)(q-s-1). \tag{2.42}$$

Thus, although the maximal degree of $Ph(G,q,s,w)$ in w is, in general, n, it is less than n if $s = 0$ or $s = 1$. In the $s = 0$ case, all dependence on w disappears (cf. (1.1)), while for $s = 1$ we have previously analyzed the maximal degree of $Ph(G,q,1,w)$ for various families of graphs in Ref. [2]. For $s = 1$, provided that G contains at least one edge, $Ph(G,q,w)$ has a factor of $(q-1)$ [2]. This is not, in general, true for s in the interval $2 \le s \le q-1$, and this is one of the ways that the properties of $Ph(G,q,s,w)$ for $2 \le s \le q-1$ differ from those for $s=1$.

One can also express $Z(G,q,s,v,w)$ as a polynomial in q with coefficients, denoted as $\alpha_{Z,G,\ell}(s,v,w)$, which are polynomials in s, v, and w:

$$Z(G,q,s,v,w) = \sum_{j=0}^{n} \alpha_{Z,G,n-j}(s,v,w) q^{n-j}. \tag{2.43}$$

Accordingly, with the notation

$$\alpha_{G,n-j}(s,w) \equiv \alpha_{Z,G,n-j}(s,-1,w), \tag{2.44}$$

we write

$$Ph(G,q,s,w) = \sum_{j=0}^{n} \alpha_{G,n-j}(s,w) q^{n-j}. \tag{2.45}$$

From our discussion above, we have the following results for these latter coefficients for $Ph(G,q,s,w)$:

$$\alpha_{G,n}(s,w) = 1 \tag{2.46}$$

and, using also (2.21),

$$\alpha_{G,n-1}(s,w) = ns(w-1) - e(R_E(G)). \tag{2.47}$$

Finally, since the variable s only enters Z via the combination $t = s(w-1)$, it is also useful to express the coefficients in (2.43) as polynomials in t, v, and w, and the coefficients in (2.45) as polynomials in t and w. For a given graph G, we find that this usually simplifies the expressions.

A chromatic polynomial $P(G,q)$, written in the form

$$P(G,q) = \sum_{j=0}^{n-k(G)} \alpha_{G,n-j} q^{n-j}, \tag{2.48}$$

has the property that the signs of the coefficients $\alpha_{G,n-j}$ alternate:

$$\mathrm{sgn}(\alpha_{G,n-j}) = (-1)^j, \quad 0 \le j \le n-k(G). \tag{2.49}$$

(where, as before, $k(G)$ denotes the number of components of G, and we shall continue, without loss of generality, to focus on connected graphs, so that $k(G)=1$). This sign alternation property can be proved by iterated application of the deletion-contraction relation. Since the weighted-set chromatic polynomial $Ph(G,q,s,w)$ does not, in general, obey a deletion-contraction relation, except for the values $w=1$, $w=0$, and $s=0$ for which it reduces to a chromatic polynomial, one does not expect the corresponding coefficients $\alpha_{G,n-j}(s,w)$ in (2.45) to have this sign-alternation property in general, and they do not. However, we have proved that if w is in the DFSCP interval $0 \le w < 1$, then the sign alternation property again holds, i.e.,

$$\mathrm{sgn}(\alpha_{G,n-j}(s,w)) = (-1)^j \quad \text{for } 0 \le w < 1 \text{ and } 0 \le j \le n-1. \tag{2.50}$$

The technical mathematical details of our proof will be given elsewhere. For the borderline cases $w=1$ and $w=0$, as well as for $s=0$ and $s=q$, $Ph(G,q,s,w)$ reduces to a chromatic polynomial, so the sign-alternation property is already established. For $j=n$, namely for the q^0 term in $Ph(G,q,s,w)$, the sign alternation also holds for $0\le w<1$; here the coefficient $\alpha_{G,0}(s,w)$ contains the factor $t=s(w-1)$ and hence vanishes at $w=1$ and $s=0$.

Setting $q=0$ in (2.6), and recalling the factorization in (2.11), we deduce that

$$Z(G,0,v,s,w)=\alpha_{Z,G,0}(s,v,w) \text{ contains a factor of } t=s(w-1). \tag{2.51}$$

The same holds, *a fortiori*, for the $v=-1$ special case, $Ph(G,0,s,w)$, i.e., $\alpha_{G,0}(s,w)$ contains a factor of $t=s(w-1)$.

From a study of chromatic polynomials, R. Read observed that the magnitudes of the coefficients of successive powers of q^{n-j}, $0\le j\le n-k(G)$ in a chromatic polynomial satisfy a unimodal property [8]. That is, the magnitudes of these coefficients get successively larger and larger, and then smaller and smaller, as j increases from 0 to $n-k(G)$. There is thus a unique maximal-magnitude coefficient, or two successive coefficients whose magnitudes are equal. From our calculations of weighted chromatic polynomials for a number of families of graphs, we have observed that in the interval $0\le w\le 1$ this property continues to hold. We therefore state the following conjecture: *Conject.* Let $Ph(G,q,s,w)$ be written as in (2.45). Then for real w in the interval $0\le w\le 1$, the quantities $(-1)^j\alpha_{G,n-j}(s,w)$, $0\le j\le n$, are positive and satisfy the unimodal property, i.e., $(-1)^j\alpha_{G,n-j}(s,w)$ get progressively larger and larger, and a maximal value is reached for a given j, or for two successive j values, and then the quantities $(-1)^j\alpha_{G,n-j}(s,w)$ get progressively smaller, as j increases from 0 to n.

2.4 Measure of Deviation from Deletion-Contraction Relation

For a graph G, let us denote the graph obtained by deleting an edge $e\in E$ as $G-e$ and the graph obtained by deleting this edge and identifying the two vertices that had been connected by it as G/e. The Potts model partition function satisfies the deletion-contraction relation (DCR)

$$Z(G,q,v)=Z(G-e,q,v)+vZ(G/e,q,v) \tag{2.52}$$

and, setting $v=-1$, the chromatic polynomial thus satisfies the DCR

$$P(G,q,v)=P(G-e,q)-P(G/e,q). \tag{2.53}$$

However, in general, neither $Z(G,q,s,v,w)$ nor $Ph(G,q,s,w)$ satisfies the respective deletion-contraction relation. For the special cases $w=1$ and $s=0$ for which $Z(G,q,s,v,w)$ and $Ph(G,q,s,w)$ reduce to $Z(G,q,v)$ and $P(G,q)$, and for the special case $w=0$ for which $Z(G,q,s,v,w)$ and $Ph(G,q,s,w)$ reduce to $Z(G,q-s,v)$ and $Ph(G,q-s)$, respectively, they do satisfy deletion-contraction relation. Hence, the deviations from such a relation for $Z(G,q,s,v,w)$ and $Ph(G,q,s,w)$ vanish in these three cases. For a given G, it is of interest to examine the quantities that measure the deviation from the DCR, namely

$$\begin{aligned}[\Delta Z(G,e,q,s,v,w)]_{DCR} &= Z(G,q,s,v,w)\\ &\quad -[Z(G-e,q,s,v,w)+vZ(G/e,q,s,v,w)]\end{aligned} \tag{2.54}$$

and

$$[\Delta Ph(G,e,q,s,w)]_{DCR}\equiv[\Delta Z(G,e,q,s,-1,w)]_{DCR}. \tag{2.55}$$

From our discussion above, it follows that $[\Delta Z(G,e,q,s,v,w)]_{DCR}=0$, and hence also $\Delta Ph(G,e,q,s,w)]_{DCR}=0$, for $w=1$, $w=0$, and $s=0$; therefore, since these functions are polynomials in these variables, they contain a factor $sw(w-1)$. Moreover, since the condition $v=0$ is equivalent to the absence of any edges, whence $Z=(q+t)^n$ and the deletion-contraction relation is satisfied trivially, $[\Delta Z(G,e,q,s,v,w)]_{DCR}$ also vanishes for $v=0$. Thus, in general,

$$[\Delta Z(G,e,q,s,v,w)]_{DCR} \text{ contains the factor } svw(w-1), \tag{2.56}$$

and $[\Delta Ph(G,e,q,s,w)]_{DCR}$ contains a factor of $sw(w-1)$. As an illustration, using our explicit calculations for n-vertex line graphs L_n and circuit graphs C_n, we find the following results. For the first two graphs, L_3 and C_3, the deletion and contraction on any edge gives the same result, so we need not specify which edge is involved. We find, for any edge e,

$$[\Delta Z(L_2,e,q,s,v,w)]_{DCR}=svw(w-1), \tag{2.57}$$

$$[\Delta Z(L_3,e,q,s,w)]_{DCR}=svw(w-1)[s(w-1)+wv+q], \tag{2.58}$$

and

$$[\Delta Z(C_3,e,q,s,v,w)]_{DCR}=svw(w-1)[wv^2+2wv+s(w-1)+q]. \tag{2.59}$$

As before, the corresponding $[\Delta Ph(G,e,q,s,w)]_{DCR}$ expressions are obtained by setting $v=-1$ in these equations. It is straightforward to calculate similar differences $[\Delta Z(G,e,q,s,v,w)]_{DCR}$ for graphs with more vertices and edges, but these are sufficient for our illustration.

2.5 Distinguishing Between Various Equivalence Classes of Graphs

An important property of the weighted-set chromatic polynomial $Ph(G,q,s,w)$ is the fact that it can distinguish between certain graphs that yield the same chromatic polynomial $P(G,q)$. This is true for all w and s values except the special values $w=1$, $w=0$, $s=0$, and $s=q$, for which $Ph(G,q,s,w)$ is reducible to a chromatic polynomial. More generally, an important property of the partition function of the Potts model in a set of nonzero external magnetic fields of the form (2.3), $Z(G,q,s,v,w)$, is that this function can distinguish between graphs that yield the same zero-field Potts model partition function, $Z(G,q,s,v,1)=Z(G,q,v)$. Two graphs G and H are defined as (i) Tutte-equivalent if they have the same Tutte polynomial, or equivalently, the same zero-field Potts model partition function, $Z(G,q,v)$, and (ii) chromatically equivalent if they have the same chromatic polynomial, $P(G,q)$. Here we recall that the Tutte polynomial $T(G,x,y)$ of a graph G is defined as

$$T(G,x,y)=\sum_{G'\subseteq G}(x-1)^{k(G')-k(G)}(y-1)^{c(G')} \tag{2.60}$$

where G' is a spanning subgraph of G and $c(G')$ denotes the number of (linearly independent) cycles in G'. (We remark that $n(G')-k(G')$ and $c(G')$ are the rank and co-rank (= cyclotomic number) of G'; thus, since $n(G')=n(G)$, $k(G')-k(G)$ is the relative rank (rank difference) of G' in G.) This polynomial is equivalent to the zero-field Potts model partition function, via the relation

$$Z(G,q,v)=(x-1)^{k(G)}(y-1)^nT(G,x,y). \tag{2.61}$$

where $y = v + 1$ as in (2.5) and

$$x = 1 + \frac{q}{v}. \tag{2.62}$$

We give some examples. Recall the definition that a tree graph is a connected graph that contains no circuits (cycles). The set of tree graphs with n vertices, generically denoted $\{T_n\}$, forms a Tutte equivalence class, with $T(T_n, x, y) = x^{n-1}$, or equivalently,

$$Z(T_n, q, v) = q(q + v)^{n-1}. \tag{2.63}$$

However, $Z(G, q, s, v, w)$ is able to distinguish between different tree graphs in a Tutte-equivalence class. Because $Z(G, q, s, v, w)$ reduces to a zero-field Potts model partition function for $w = 1$, $w = 0$, $s = 0$, and $s = q$, it follows that the difference between $Z(G, q, s, v, w)$ and $Z(H, q, s, v, w)$ for two Tutte-equivalent graph G and H must vanish if $w = 1$, $w = 0$, $s = 0$, or $s = q$. This difference also vanishes for $v = 0$, because in this case the only spanning subgraph that contributes to the sum in (2.6) is the one with no edges, which is the same for any (connected) Tutte-equivalent G and H. Since these are all polynomials, it thus follows that

$$Z(G, q, s, v, w) - Z(H, q, s, v, w) \text{ contains the factor } s(q - s)vw(w - 1). \tag{2.64}$$

As an illustration, using our results for $Z(L_4, q, s, v, w)$ and $Z(S_4, q, s, v, w)$, we have

$$Z(S_4, q, s, v, w) - Z(L_4, q, s, v, w) = s(q - s)v^2 w(w - 1)^2 \tag{2.65}$$

and consequently

$$Ph(S_4, q, s, w) - Ph(L_4, q, s, w) = s(q - s)w(w - 1)^2. \tag{2.66}$$

The zero-field Potts model partition function $Z(G, q, v)$, or equivalently, the Tutte polynomial $T(G, x, y)$, encodes information on the number of (linearly independent) cycles contained in the graph G, as is evident from the definition (2.60). Define two scaled variables as

$$q' \equiv \frac{q}{s}, \qquad v' \equiv \frac{v}{s}. \tag{2.67}$$

Let us consider a graph, denoted G_{nc}, which contains no cycles (nc), i.e., which has $c(G) = 0$. A connected graph of this type is a tree graph, while a general graph is called a forest. For a graph G_{nc} we find the following scaling relation:

$$Z(G_{nc}, q, s, v, w) = s^n Z(G_{nc}, q', 1, v', w). \tag{2.68}$$

This is proved as follows. We start with the cluster formula (2.6) and rewrite this as

$$Z(G_{nc}, q, s, v, w) = \sum_{G' \subseteq G_{nc}} (v')^{e(G')} s^{e(G')+k(G')} \prod_{i=1}^{k(G')} \left(q' - 1 + w^{n(G'_i)}\right). \tag{2.69}$$

We next use the relation, which holds for any graph G',

$$c(G') + n(G') = e(G') + k(G') \tag{2.70}$$

and the fact that $n(G') = n(G) \equiv n$ to rewrite the factor $s^{e(G')+k(G')}$ as $s^{c(G')+n}$. Since G_{nc} has no cycles, it follows that the same is true for any subgraph of G_{nc}, in particular, the spanning subgraph G', so $c(G') = 0$. Hence, we can move the factor of s^n in front of the summation, and we have

$$\begin{aligned} Z(G_{nc}, q, s, v, w) &= s^n \sum_{G' \subseteq G_{nc}} (v')^{e(G')} \prod_{i=1}^{k(G')} \left(q' - 1 + w^{n(G'_i)}\right) \\ &= s^n Z(G_{nc}, q', 1, v', w). \qquad \square \end{aligned} \tag{2.71}$$

Hence, the difference

$$[\Delta Z(G, q, s, v, w)]_{cycles} = Z(G, q, s, v, w) - s^n Z(G, q', 1, v', w) \tag{2.72}$$

provides a measure of the number of cycles in G. For example, using our general result for $Z(C_n, q, s, v, w)$ given below in (5.2), we calculate

$$[\Delta Z(C_n, q, s, v, w)]_{cycles} = \frac{(s-1)(q-s+sw^n)v^n}{s}. \tag{2.73}$$

Moreover, the proper q-coloring condition implies that if two different graphs G and H differ only in having different numbers of edges connecting a pair of adjacent vertices, for one or more such pairs, so that $R_E(G) = R_E(H)$, then $Ph(G, q, s, w) = Ph(H, q, s, w)$. A simple example is provided by the line and circuit graphs with $n = 2$ vertices, L_2 and C_2, the latter of which has a double edge connecting the two vertices. Using our results in (3.2) and (5.7), we calculate the difference

$$Z(C_2, q, s, v, w) - Z(L_2, q, s, v, w) = v(v+1)\left[q + s(s-1)(w+1)\right]. \tag{2.74}$$

The fact that the difference in (2.74) vanishes for $v = -1$, i.e., that $Ph(L_2, q, s, w) = Ph(C_2, q, s, w)$, is a special case of the general result (2.21).

In the context of graph coloring, since $s \in I_s$, if one sets q to a particular value, this implicitly sets a corresponding upper bound on s. In particular, if $q = 1$, then s can take on only the values 0 or 1, and these are related by the symmetry (2.13). For $s = 0$, we have, by (2.15), that

$$Z(G, 1, 0, v, w) = Z(G, 1, v) = y^{e(G)}, \tag{2.75}$$

where $y = v + 1$. For $s = 1$, applying (2.13), we have

$$Z(G, 1, 1, v, w) = w^n Z(G, 1, 0, v, w^{-1}) = w^n Z(G, 1, v) = y^{e(G)} w^n. \tag{2.76}$$

If G has at least one edge, then the right-hand sides of both (2.75) and (2.76) vanish for the case $y = 0$ ($v = -1$) that yields the weighted-set chromatic polynomial. In order for two graphs G and H to be chromatically equivalent, a necessary condition is that they must have the same number of vertices, $n(G) = n(H)$, since the degree in q of $P(G, q)$ is $n(G)$. An elementary property of the chromatic polynomial $P(G, q)$, proved by iterative application of the deletion-contraction theorem, is that the coefficient of the $q^{n(G)-1}$ term is $-e(R_E(G))$. Therefore, another necessary condition that two graphs G and H must satisfy in order to be chromatically equivalent is that $e(R_E(G)) = e(R_E(H))$. If G contains at least one edge, then $Ph(G, 1, s, w) = 0$. Note here that since s is bounded above by q, it follows that if $q = 1$, then s can only take on the values $s = 0$ or $s = 1$. Hence, if G and H are chromatically equivalent, then either (i) neither contains any edges, in which case $Ph(G, q, s, w) = Ph(H, q, s, w) = (q + t)^n$, where $n = n(G) = n(H)$, or (ii) if G, and hence H, contains at least one edge, then $Ph(G, 1, s, w) = Ph(H, 1, s, w) = 0$. Hence, if G and H are chromatically equivalent and contain at least one edge, then the difference $Ph(G, q, s, w) - Ph(H, q, s, w)$ contains a factor that vanishes when $q = 1$. Because of the implicit condition on s for a given q, this factor is not, in general, $(q - 1)$. As an example, the difference $Ph(S_4, q, s, w) - Ph(L_4, q, s, w)$ in (2.66) contains the factor $s(q - s)$. For $q = 1$, the values of s are implicitly restricted to $s = 0$ and $s = 1$. For either of these choices, the factor, and hence the difference, vanishes.

A remark concerning duality for planar graphs is also in order here. Let $G = (V, E)$ be a planar graph, and denote its planar dual by G^*. The chromatic polynomial $P(G, q)$ counts not just the proper q-colorings of the vertices of G, but also, and equivalently, the proper q colorings of the faces of G^*. Similarly, for this planar graph G, the weighted-set chromatic polynomial $Ph(G, q, s, w)$ describes not just the weighted-set proper q-colorings of the vertices of G but also, and equivalently, the weighted-set proper q-colorings of the faces of G^*.

2.6 Lower Bounds on $Ph(G, q, s, w)$

We derive some bounds on $Ph(G, q, s, w)$ for certain types of graphs. Our method for this will be to calculate the contribution to $Ph(G, q, s, w)$ resulting from a certain procedure for performing proper q-colorings of the graph G. By the general formula expressing $Ph(G, q, s, w)$ as the $v = -1$ special case of the Potts model partition function $Z(G, q, s, v, w)$ together with the formulation of this partition function as a sum over spin (or equivalently, color) configurations, (2.1) with (2.2), it follows that there are other color configurations in addition to the particular one that we consider, contributing (positive terms) to $Ph(G, q, s, w)$. Therefore each particular proper q-coloring procedure that we consider yields a lower bound on $Ph(G, q, s, w)$. The specific proper q-coloring procedure that provides a good lower bound to $Ph(G, q, s, w)$ depends on the type of graph G, the values of q, s, and w.

Let us consider a bipartite graph G_{bip}, defined as a graph whose vertex set V can be partitioned into subsets V_1 and V_2 such that a vertex in V_1 has edges that connect it only to a vertex or vertices in V_2 and vice versa. An equivalent condition for a graph to be bipartite is that its chromatic number $\chi(G_{bip}) = 2$. As above, we denote the number of vertices in G, as $n(G) \equiv n$ and, further, the number of vertices in V_1 and V_2 as n_1 and n_2, respectively. With no loss of generality, we label these subsets of vertices so that $n_1 \leq n_2$. These numbers n_1 and n_2 may be comparable or may be quite different. For example for a lattice graph such as the circuit graph, the square, honeycomb, simple cubic, or body-centered cubic lattices, with periodic boundary conditions that preserve the bipartite nature of the lattices, $n_1 = n_2$. However, for the star graph S_n, V_1 consists of the central vertex, so that $n_1 = 1$, while V_2 is comprised of all of the vertices on the ends of the edges forming the rays of the star, so $n_2 = n - 1$. For an S_n graph with $n \gg 1$, it follows that $n_2 \gg n_1$.

For the following, we assume that $q \geq 2$ so that a proper q-coloring of the bipartite graph G_{bip} is possible. If $w = 1$, then a well-known elementary lower bound on $P(G_{bip}, q)$ is obtained by (i) assigning a single color to all of the vertices in V_1 and (ii) independently choosing a color out of the remaining $q - 1$ for each of the vertices in V_2. There are $q(q - 1)^{n_2}$ ways of doing this. Since, in general, there are also other color configurations contributing to a proper q-coloring of G_{bip}, this yields the lower bound

$$P(G_{bip}, q) \geq q(q-1)^{n_2}. \tag{2.77}$$

For the weighted-set chromatic polynomial, the situation is more complicated. With no loss of generality, we again label the vertex subsets so that $n_1 \leq n_2$. We also take $q \geq 2$ so that a proper (weighted) q-coloring is possible and also assume that $s \geq 2$ (and $s \leq q$, as discussed above). Then for sufficiently large $w > 1$ in the FSCP interval, one lower bound on $Ph(G_{bip}, q, s, w)$ is obtained by (i) assigning one color from the favorably weighted set I_s to all of the vertices in V_1, and then (ii) independently, for each vertex in V_2, assigning a color from among the remaining $s-1$ colors in I_s. This combined color assignment can be made in $s(s-1)^{n_2}$ ways, and yields a contribution $s(s-1)^{n_2}w^n$ to $Ph(G_{bip}, q, s, w)$. From the inequality (2.20), it then follows that

$$Ph(G_{bip}, q, s, w) \geq s(s-1)^{n_2}w^n. \tag{2.78}$$

(Since $n_1 \leq n_2$, this is an equivalent or better bound than the one obtained by making the above color assignments with V_1 and V_2 reversed, viz., $Ph(G_{bip}, q, s, w) \geq s(s-1)^{n_1}w^n$.)

On the other hand, for w in the DFSCP interval $0 \leq w < 1$, it can be preferable to minimize the number of vertices with colors in I_s in order to maximize the contribution to $Ph(G_{bip}, q, s, w)$. For sufficiently small (positive) w, provided that $q \geq s+2$, a lower bound on $Ph(G_{bip}, q, s, w)$ is then obtained by (i) assigning a single color from $I_s^{\perp}$ to all of the vertices of V_1, and (ii) independently, for each vertex of V_2, assigning a color from among the remaining $(q-s-1)$ colors in $I_s^{\perp}$. This combined color assignment can be made in $(q-s)(q-s-1)^{n_2}$ ways. Invoking the inequality (2.20) again, we have

$$Ph(G_{bip}, q, s, w) \geq (q-s)(q-s-1)^{n_2}. \tag{2.79}$$

If w is only slightly less than unity, say $1-\epsilon < w < 1$ for sufficiently small positive ϵ, provided also that $q \geq s+1$, a different type of lower bound on $Ph(G_{bip}, q, s, w)$ can be obtained by the following proper q-coloring procedure: (i) one assigns a single color from I_s to all of the vertices of V_1 and (ii) independently for each vertex in V_2, one assigns a color from $I_s^{\perp}$. There are $s(q-s)^{n_2}$ ways of doing this, and the resultant term in $Ph(G_{bip}, q, s, w)$ is $sw^{n_1}(q-s)^{n_2}$. Using the inequality (2.20) again, one thus infers the lower bound

$$Ph(G_{bip}, q, s, w) \geq sw^{n_1}(q-s)^{n_2}. \tag{2.80}$$

Which of these lower bounds is the best depends in detail on G_{bip} (in particular, on n_1 and n_2), q, s, and w. It is easy to generalize these lower bounds to multipartite graphs.

3 Calculations of $Z(G, q, s, v, w)$ and $Ph(G, q, s, w)$ for Some Families of Graphs

In this section we give some illustrative explicit calculations of $Z(G, q, s, v, w)$ and $Ph(G, q, s, w)$ for various families of graphs. Although we generally consider connected graphs, we note that for the graph N_n consisting of n vertices with no edges,

$$Z(N_n, q, s, v, w) = Ph(N_n, q, s, w) = (q+t)^n. \tag{3.1}$$

3.1 Line Graphs L_n

The line graph (also called path graph) L_n is the graph consisting of n vertices with each vertex connected to the next one by one edge. In general, $\alpha_{Z,L_n,n-1}(q,s,v) = nt + (n-1)v$. We proceed to give some explicit results for $Z(L_n,q,s,w)$ for various values of n. The case $L_1 = N_1$ is already covered by (3.1). For the first few n values, we also give the expansions in terms of powers of q and, for this latter expansion, we use the variables q, t, and w instead of q, s, and w, because this makes the expressions shorter:

$$\begin{aligned} Z(L_2,q,s,v,w) &= s(s+v)w^2 + 2s(q-s)w + (q-s)(q-s+v) \\ &= q^2 + (2t+v)q + t[t + v(w+1)], \end{aligned} \tag{3.2}$$

$$\begin{aligned} Z(L_3,q,s,v,w) &= s(s+v)^2w^3 + s(q-s)(3s+2v)w^2 \\ &\quad + s(q-s)[3(q-s)+2v]w + (q-s)(q-s+v)^2 \\ &= q^3 + (3t+2v)q^2 + (3t^2 + 2vtw + 4vt + v^2)q \\ &\quad + t(v^2w^2 + 2vtw + wv^2 + t^2 + 2vt + v^2). \end{aligned} \tag{3.3}$$

For L_4 we give only the expansion in powers of w, since the equivalent expansion in powers of q becomes somewhat lengthy:

$$\begin{aligned} Z(L_4,q,s,v,w) &= s(s+v)^3w^4 + 2s(q-s)(s+v)(2s+v)w^3 \\ &\quad + s(q-s)[-3(s^2+(q-s)^2) + 3q(q+v) + 2v^2]w^2 \\ &\quad + 2s(q-s)(q-s+v)[2(q-s)+v]w + (q-s)(q-s+v)^3. \end{aligned} \tag{3.4}$$

3.2 Star Graphs S_n

A star graph S_n consists of one central vertex with degree $n-1$ connected by edges with $n-1$ outer vertices, each of which has degree 1. (The context will always make clear the difference between this symbol for the n-vertex star graph and the symbol S_n for the symmetric group on n objects.) The graph S_2 is degenerate in the sense that it has no central vertex but instead coincides with L_2. The graph S_3 is nondegenerate, and coincides with L_3, while the S_n for $n \ge 4$ are distinct graphs not coinciding with those of other families. For $n \ge 2$, the chromatic number is $\chi(S_n) = 2$. By the use of combinatoric coloring methods, we have derived the following general formula for $Z(S_n,q,s,v,w)$:

$$\begin{aligned} Z(S_n,q,s,v,w) &= \sum_{j=0}^{n-1} \binom{n-1}{j} v^j (\tilde{q} + sw^{j+1})(\tilde{q} + sw)^{n-1-j} \\ &= (q-s)[q + s(w-1) + v]^{n-1} + sw[q + s(w-1) + wv]^{n-1} \end{aligned} \tag{3.5}$$

where $\tilde{q} = q - s$, as given in (2.22). Evaluating (3.5) for $v = -1$ yields $Ph(S_n,q,s,w)$. As an explicit example, for the graph S_4, we calculate

$$\begin{aligned} Z(S_4,q,s,v,w) &= Z(T_4,s,v)w^4 + s(q-s)(4s^2 + 6sv + 3v^2)w^3 \\ &\quad + 3s(q-s)[2s(q-s) + qv]w^2 \\ &\quad + s(q-s)[4(q-s)^2 + 6(q-s)v + 3v^2]w \\ &\quad + Z(T_4, q-s, v) \end{aligned} \tag{3.6}$$

where $Z(T_n,q,v)$ was given in (2.63).

3.3 Complete Graphs K_n

The complete graph K_n is the graph with n vertices such that each vertex is connected to every other vertex by one edge. The chromatic number is $\chi(K_n) = n$ and the number of edges is $e(K_n) = \binom{n}{2}$. Let us introduce the compact notation $x_\theta \equiv x\theta(x)$, where $\theta(x)$ is the step function from $\mathbb{R} \to \{0, 1\}$ defined as $\theta(x) = 1$ if $x > 0$ and $\theta(x) = 0$ if $x \le 0$. We have derived the following theorem giving a general formula for $Ph(K_n, q, s, w)$:

$$Ph(K_n, q, s, w) = \sum_{\ell=0}^{n} \beta_{K_n,\ell}(q, s) w^\ell \tag{3.7}$$

where

$$\beta_{K_n,\ell} = \binom{n}{\ell} \left[\prod_{j=0}^{(\ell-1)_\theta} (s-j) \right] \left[\prod_{m=0}^{(n-\ell-1)_\theta} (q-s-m) \right]. \tag{3.8}$$

Proof This result is proved by a combinatoric coloring argument. Accordingly, we take q to be a non-negative integer. The resultant (3.7) and (3.8) allow the extension of q to $\mathbb{R}$ (and $\mathbb{C}$). First, if $q < \chi(K_n) = n$, then $Ph(K_n, q, s, w)$ vanishes identically. Hence, we shall formally take $q \ge n$ to begin with; once we have obtained the results (3.7) and (3.8), it will be seen that they allow an extension of q away from this range. If $s \ge n$, then one can assigning n different colors to the n vertices of K_n from the set I_s, and this gives rise to a term with degree n in w. To determine the coefficient of this term, we enumerate the number of ways this color assignment can be made. We pick a given vertex and assign some color from I_s to this vertex, which we can do in any of s ways. Then we go on to the next vertex and assign one of the remaining $s - 1$ colors in I_s to that vertex, and so on for the n vertices. The number of ways of making this color assignment, i.e., the coefficient of the term in $Ph(K_n, q, s, w)$ of maximal degree in w, viz., w^n, is therefore

$$\beta_{K_n,n}(q, s) = \prod_{j=0}^{n-1} (s-j) = P(K_n, s). \tag{3.9}$$

The fact that this coefficient is $P(K_n, s)$ agrees with the $v = -1$ special case of the general result of (2.35). Similarly, the term of order w^0 is obtained by assigning n different colors to the n vertices of K_n from the orthogonal set $S^\perp$. By reasoning analogous to that given above, it follows that the number of ways of doing this is given by replacing s by $q - s$ in (3.9), so

$$\beta_{K_n,0}(q, s) = \prod_{j=0}^{n-1} (q-s-j) = P(K_n, q-s). \tag{3.10}$$

Having illustrated the logic on these two extremal terms, let us next consider the general w^ℓ term with $0 \le \ell \le n$. This term arises from color assignments in which we pick ℓ different colors from the set I_s and assign them to ℓ of the n vertices of K_n, and then $n - \ell$ different colors from the orthogonal complement set $S^\perp$, which are assigned to the remaining $n - \ell$ vertices. The number of ways of doing this is

$$\beta_{K_n,\ell} = \left[\prod_{j=0}^{(\ell-1)_\theta} (s-j) \right] \left[\prod_{m=0}^{(n-\ell-1)_\theta} (q-s-m) \right]. \tag{3.11}$$

This proves the result in (3.7) and (3.8). □

Evidently, with the polynomial $Ph(K_n, q, s, w)$ as specified in these equations, one can extend q and s away from non-negative integer values. Our result in (3.7) and (3.8) generalizes the result for the case $s = 1$ given in [2]. As is evident, for $w = 1$ or $s = 0$, $Ph(K_n, q, s, w)$ reduces to the (usual, unweighted) chromatic polynomial

$$P(K_n, q) = \prod_{j=0}^{n-1} (q - j). \tag{3.12}$$

A corollary of (3.7) and (3.8) is that

$$\text{If } s < n, \text{ then } \beta_{K_n,j}(q, s) = 0 \text{ for } s < j \le n \tag{3.13}$$

and hence

$$\deg_w(Ph(K_n, q, s, w)) = \min(n, s). \tag{3.14}$$

Having calculated $Ph(K_n, q, s, w)$, it is appropriate to discuss here another aspect in which the weighted-set chromatic polynomial differs from the (usual, unweighted) chromatic polynomial. Let us consider a graph G that has the property of being composed of the union of two subgraphs, $G = G_1 \cup G_2$, such that $G_1 \cap G_2 = K_m$ for some m. In the rest of this paragraph, we assume that G has this property. Then $P(G, q)$ satisfies the relation

$$P(G, q) = \frac{P(G_1, q) P(G_2, q)}{P(K_m, q)}. \tag{3.15}$$

(This is sometimes called the complete-graph intersection theorem (KIT) for chromatic polynomials.) In contrast, in general, $Ph(G, q, s, w)$ is not equal to $Ph(G_1, q, s, w) \times Ph(G_2, q, s, w)/Ph(K_m, q, s, w)$. This equality holds only for the four values $w = 1$, $w = 0$, $s = 0$, and $s = q$ where $Ph(G, q, s, w)$ reduces to a chromatic polynomial. As a measure of the deviation from equality, we define

$$[\Delta Ph(G, q, s, w)]_{KIT} \equiv Ph(G, q, s, w) - \frac{Ph(G_1, q, s, w) Ph(G_2, q, s, w)}{Ph(K_m, q, s, w)}. \tag{3.16}$$

The vanishing of $[\Delta Ph(G, q, s, w)]_{KIT}$ for $w = 1$, $w = 0$, and $s = 0$ is obvious. To show that this vanishes for $s = q$, we use the relation (2.18) and obtain

$$[\Delta Ph(G, q, q, w)]_{KIT} = w^n \left[P(G, q) - \frac{P(G_1, q) P(G_2, q)}{P(K_m, q)} \right] = 0. \tag{3.17}$$

Combining these results with the property that $[\Delta Ph(G, q, s, w)]_{KIT}$ is a rational function in its arguments, we have thus shown that

$$[\Delta Ph(G, q, s, w)]_{KIT} \text{ contains the factor } s(q - s)w(w - 1). \tag{3.18}$$

We give two illustrations. The line graph L_3 has the property of being comprised of two L_2 graphs intersecting on $L_1 = K_1$. Using (the $v = -1$ special cases of) our results in (3.1), (3.2), and (3.4), we calculate

$$[\Delta Ph(L_3, q, s, w)]_{KIT} = \frac{s(q - s)w(w - 1)^2}{q + s(w - 1)}. \tag{3.19}$$

Similarly, the graph L_4 can be decomposed into L_3 and L_2 subgraphs that intersect on an $L_1 = K_1$ graph. Using our results in (3.1) and (3.2)–(3.4), we calculate

$$[\Delta Ph(L_4, q, s, w)]_{KIT} = \frac{s(q-s)w(w-1)^2[q+s(w-1)-(w+1)]}{q+s(w-1)}. \tag{3.20}$$

A slightly more complicated case is the 4-vertex graph C_{4d} consisting of a box with one diagonal edge added. This graph has the structure of two $C_3 = K_3$ subgraphs intersecting on the diagonal edge graph, $L_2 = K_2$. For this graph we calculate

$$[\Delta Ph(C_{4d}, q, s, w)]_{KIT} = 2s(q-s)w(w-1)^2\left[1 - \frac{2(q-1)w}{Ph(K_2, q, s, w)}\right]. \tag{3.21}$$

We see that each of these differences $[\Delta Ph(G, q, s, w)]_{KIT}$ satisfies the general factorization property of (3.18).

4 $Z(G, q, s, v)$ and $Ph(G, q, s, w)$ for Cyclic Strip Graphs

4.1 General Structure

References [1, 3] have given a general structural formula for $Z(G_s, L_y \times m, BC, q, s, v, w)$ on strip graphs G_s of width L_y vertices and length L_x, with cyclic or Möbius boundary conditions (BC's) for the case $s = 1$. Here we discuss the generalizations to arbitrary (integer) s in the interval $0 \le s \le q$. For cyclic strip graphs G_s of this type we have

$$Z(G_s, L_y \times m, cyc, q, s, v, w) = \sum_{d=0}^{L_y} \tilde{c}^{(d)} \sum_{j=1}^{n_{Zh}(L_y,d,s)} [\lambda_{Z,G_s,L_y,d,j}(q, s, v, w)]^m, \tag{4.1}$$

where $m = L_x$ for strips of the square and triangular lattices and $m = L_x/2$ for strips of the honeycomb lattice. The coefficients $\tilde{c}^{(d)}$ are given by

$$\tilde{c}^{(d)} \equiv c^{(d)}(\tilde{q}) = \sum_{j=0}^{d} (-1)^j \binom{2d-j}{j} \tilde{q}^{d-j} \tag{4.2}$$

where $\tilde{q} = q - s$, as specified in (2.22) (see also [19, 20]). The first few of these coefficients are $\tilde{c}^{(0)} = 1$, $\tilde{c}^{(1)} = \tilde{q} - 1 = q - s - 1$, $\tilde{c}^{(2)} = \tilde{q}^2 - 3\tilde{q} + 1$, etc. For Möbius strips, the switching of certain $\tilde{c}^{(d)}$'s, as specified for $s = 1$ in general in [3, 19], generalizes to arbitrary s in the interval $0 \le s \le q$. Further, from (4.8), it follows that there is only one term with $d = L_y$, and we find (dropping the j subscript)

$$\lambda_{Z,L_y,L_y}(q, s, v, w) = v. \tag{4.3}$$

For $Ph(G_s, L_y \times m, cyc, q, s, w) = Z(G, L_y \times m, cyc, q, s, -1, w)$, we have

$$Ph(G, L_y \times m, cyc, q, s, w) = \sum_{d=0}^{L_y} \tilde{c}^{(d)} \sum_{j=1}^{n_{Ph}(L_y,d,s)} [\lambda_{Ph,G_s,L_y,d,j}(q, s, w)]^m. \tag{4.4}$$

As with Z, there is only one term with $d = L_y$, and (4.3) shows that this is

$$\lambda_{Ph,G_s,L_y,L_y}(q,s,w) = -1. \tag{4.5}$$

The $n_{Zh}(L_y,d,s)$ satisfy the identity

$$\sum_{d=0}^{L_y} \tilde{c}^{(d)} n_{Zh}(L_y,d,s) = q^{L_y} \tag{4.6}$$

while the $n_{Ph}(L_y,d,s)$ satisfy the identity

$$\sum_{d=0}^{L_y} \tilde{c}^{(d)} n_{Ph}(L_y,d,s) = P(L_y,q) = q(q-1)^{L_y-1}. \tag{4.7}$$

The reason why these identities hold for general s with the same right-hand side as for $s = 0$ and $s = 1$ is that the basic coloring constraints remain the same; the only thing that is different for nonzero s is the weighting factors. One method of calculating the $n_{Zh}(L_y,d,s)$ and $n_{Ph}(L_y,d,s)$ is to differentiate these respective equations L_y times. One thereby obtains two respective sets of $L_y + 1$ linear equations in the $L_y + 1$ unknowns $n_{Zh}(L_y,d,s)$ and $n_{Ph}(L_y,d,s)$ for $d = 0, 1, \dots, L_y$. Solving these equations determines these numbers $n_{Zh}(L_y,d,s)$ and $n_{Ph}(L_y,d,s)$.

We have used the method above to calculate the $n_{Zh}(L_y,d,s)$ and $n_{Ph}(L_y,d,s)$. For the $n_{Zh}(L_y,d,s)$ we find

$$n_{Zh}(L_y,L_y,s) = 1, \tag{4.8}$$

$$n_{Zh}(L_y,L_y-1,s) = (s+1)L_y + (L_y-1), \tag{4.9}$$

$$n_{Zh}(L_y+1,0,s) = (s+1)n_{Zh}(L_y,0,s) + n_{Zh}(L_y,1,s) \tag{4.10}$$

and, for $1 \le d \le L_y + 1$,

$$\begin{aligned} n_{Zh}(L_y+1,d,s) &= n_{Zh}(L_y+1,d-1,s) + (s+2)n_{Zh}(L_y,d,s) \\ &\quad + n_{Zh}(L_y,d+1,s). \end{aligned} \tag{4.11}$$

Some additional results, besides the general formulas for $n_{Zh}(L_y,d,s)$ with $d = L_y$ and $d = L_y - 1$ given in (4.8) and (4.9), are

$$L_y = 2: \quad n_{Zh}(2,0,s) = s^2 + 2s + 2, \tag{4.12}$$

$$\begin{aligned} L_y = 3: \quad & n_{Zh}(3,0,s) = s^3 + 3s^2 + 6s + 5, \\ & n_{Zh}(3,1,s) = 3(s^2 + 3s + 3), \end{aligned} \tag{4.13}$$

$$\begin{aligned} L_y = 4: \quad & n_{Zh}(4,0,s) = s^4 + 4s^3 + 12s^2 + 20s + 14, \\ & n_{Zh}(4,1,s) = 4s^3 + 18s^2 + 36s + 28, \\ & n_{Zh}(4,2,s) = 6s^2 + 20s + 20. \end{aligned} \tag{4.14}$$

The numbers $n_{Zh}(L_y,d,s)$ of $\lambda_{Z,L_y,d,j}(q,s,v,w)$'s corresponding to each $\tilde{c}^{(d)}$ in the general Potts model partition function are reduced for the special case $v = -1$ that yields the weighted-set chromatic polynomial. By coloring combinatoric arguments similar to those used in [19] and [3], we determine the $n_{Ph}(L_y,d,s)$ as follows. The numbers $n_{Ph}(L_y,d,s)$ are identically zero for $d>L_y$, and

$$n_{Ph}(L_y,L_y,s)=1, \tag{4.15}$$

$$n_{Ph}(L_y,L_y-1,s)=(s+1)L_y, \tag{4.16}$$

$$n_{Ph}(L_y+1,0,s)=sn_{Ph}(L_y,0,s)+n_{Ph}(L_y,1,s) \tag{4.17}$$

and, for $1\le d\le L_y+1$,

$$\begin{aligned} n_{Ph}(L_y+1,d,s) &= n_{Ph}(L_y+1,d-1,s)+(s+1)n_{Ph}(L_y,d,s)\\ &\quad + n_{Ph}(L_y,d+1,s). \end{aligned} \tag{4.18}$$

Some additional results, besides the general formulas for $n_{Ph}(L_y,d,s)$ for $d=L_y$ and $d=L_y-1$ given in (4.15) and (4.16), are

$$L_y=2{:}\quad n_{Ph}(2,0,s)=s^2+s+1, \tag{4.19}$$

$$\begin{aligned} L_y=3{:}\quad & n_{Ph}(3,0,s)=s^3+s^2+3s+2,\\ & n_{Ph}(3,1,s)=3s^2+5s+4, \end{aligned} \tag{4.20}$$

$$\begin{aligned} L_y=4{:}\quad & n_{Ph}(4,0,s)=s^4+s^3+6s^2+7s+4,\\ & n_{Ph}(4,1,s)=4s^3+9s^2+15s+9,\\ & n_{Ph}(4,2,s)=6s^2+11s+8. \end{aligned} \tag{4.21}$$

As we have noted above, for $s=0$, $Z(G,q,s,v,w)$ reduces to $Z(G,q,v)$, so we focus on the nonzero (integer) s values in the set I_s. For these values we find

$$n_{Ph}(L_y,d,s)=n_{Zh}(L_y,d,s-1)+n_{Zh}(L_y-1,d,s-1). \tag{4.22}$$

From our determination of the numbers $n_{Zh}(L_y,d,s)$ and $n_{Ph}(L_y,d,s)$, we next calculate the total numbers

$$N_{Zh,L_y,s}=\sum_{d=0}^{L_y} n_{Zh}(L_y,d,s) \tag{4.23}$$

and

$$N_{Ph,L_y,s}=\sum_{d=0}^{L_y} n_{Ph}(L_y,d,s). \tag{4.24}$$

From (4.22) it follows that for nonzero (integer) s values in the set I_s,

$$N_{Ph,L_y,s}=N_{Zh,L_y,s-1}+N_{Zh,L_y-1,s-1}. \tag{4.25}$$

Using our results in (4.8)–(4.11), we find that, for nonzero $s\in I_s$,

$$N_{Zh,L_y,s}=\sum_{j=0}^{L_y}\binom{L_y}{j}\binom{2j}{j}s^{L_y-j} \tag{4.26}$$

and hence, using (4.25), we have

$$N_{Ph,L_y,s} = \sum_{j=0}^{L_y} \binom{L_y}{j}\binom{2j}{j}(s-1)^{L_y-j} + \sum_{j=0}^{L_y-1} \binom{L_y-1}{j}\binom{2j}{j}(s-1)^{L_y-1-j}. \tag{4.27}$$

A few explicit results for low values of s are

$$N_{Zh,1,s} = s+2, \tag{4.28}$$

$$N_{Zh,2,s} = s^2+4s+6, \tag{4.29}$$

$$N_{Zh,3,s} = s^3+6s^2+18s+20 = (s+2)(s^2+4s+10), \tag{4.30}$$

$$N_{Zh,4,s} = s^4+8s^3+36s^2+80s+70, \tag{4.31}$$

$$N_{Zh,5,s} = s^5+10s^4+60s^3+200s^2+350s+252 = (s+2)(s^4+8s^3+44s^2+112s+126), \tag{4.32}$$

$$N_{Zh,6,s} = s^6+12s^5+90s^4+400s^3+1050s^2+1512s+924, \tag{4.33}$$

and

$$N_{Ph,1,s} = s+2, \tag{4.34}$$

$$N_{Ph,2,s} = s^2+3s+4, \tag{4.35}$$

$$N_{Ph,3,s} = s^3+4s^2+11s+10, \tag{4.36}$$

$$N_{Ph,4,s} = s^4+5s^3+21s^2+37s+26, \tag{4.37}$$

$$N_{Ph,5,s} = s^5+6s^4+34s^3+88s^2+123s+70, \tag{4.38}$$

and

$$N_{Ph,6,s} = s^6+7s^5+50s^4+170s^3+366s^2+401s+192. \tag{4.39}$$

For $L_y \gg 1$, these total numbers have the following dominant asymptotic exponential growth rates (suppressing power-law prefactors):

$$N_{Zh,L_y,s} = (s+4)^{L_y} \quad \text{for } L_y \to \infty \tag{4.40}$$

and

$$N_{Ph,L_y,s} = (s+3)^{L_y} \quad \text{for } L_y \to \infty. \tag{4.41}$$

We note that the $s \to q-s$ symmetry (2.13) is not manifestly evident in the various results that we have given for $n_{Zh}(L_y,d,s)$, $n_{Ph}(L_y,d,s)$, $N_{Zh,L_y,s}$, and $N_{Ph,L_y,s}$. This symmetry arises via identities involving the $\tilde{c}^{(d)}$ and the λ's. The symmetry (2.13) implies that (for $s \in I_s$ and denoting $L_x \equiv m$)

$$\sum_{d=0}^{L_y} c^{(d)}(q-s) \sum_{j=1}^{n_{Zh}(L_y,d,s)} [\lambda_{Z,L_y,d,j}(q,s,v,w)]^m = w^n \sum_{d=0}^{L_y} c^{(d)}(s) \sum_{j=1}^{n_{Zh}(L_y,d,q-s)} [\lambda_{Z,L_y,d,j}(q,q-s,v,w^{-1})]^m \tag{4.42}$$

and, for $v=-1$,

$$\sum_{d=0}^{L_y} c^{(d)}(q-s) \sum_{j=1}^{n_{Ph}(L_y,d,s)} [\lambda_{Ph,L_y,d,j}(q,s,w)]^m = w^n \sum_{d=0}^{L_y} c^{(d)}(s) \sum_{j=1}^{n_{Ph}(L_y,d,q-s)} [\lambda_{Ph,L_y,d,j}(q,q-s,w^{-1})]^m. \tag{4.43}$$

In particular, when $s=q$, Z simplifies to a multiple times the zero-field Z as specified by the general relation (2.18), with a consequent reduction in the number of distinct $\lambda_{Z,L_y,d,j}(q,s,v,w)$'s. Using the fact that $c^{(d)}(0) = (-1)^d$ [19], we can express this reduction as

$$\sum_{d=0}^{L_y} (-1)^d \sum_{j=1}^{n_{Zh}(L_y,d,q)} [\lambda_{Z,L_y,d,j}(q,q,v,w)]^m = w^n \sum_{d=0}^{L_y} c^{(d)}(q) \sum_{j=1}^{n_Z(L_y,d)} [\lambda_{Z,L_y,d,j}(q,v)]^m \tag{4.44}$$

where $n_Z(L_y,d) \equiv n_{Zh}(L_y,d,0)$. For $s=q-1$, (4.42) yields the identity

$$\sum_{d=0}^{L_y} c^{(d)}(1) \sum_{j=1}^{n_{Zh}(L_y,d,q-1)} [\lambda_{Z,L_y,d,j}(q,q-1,v,w)]^m = w^n \sum_{d=0}^{L_y} c^{(d)}(q-1) \sum_{j=1}^{n_{Zh}(L_y,d,1)} [\lambda_{Z,L_y,d,j}(q,1,v,w^{-1})]^m \tag{4.45}$$

where $c^{(d)}(1)$ takes on the values [19]

$$c^{(d)}(1) = \begin{cases} 1 & \text{if } d = 0 \mod 3 \\ 0 & \text{if } d = 1 \mod 3 \\ -1 & \text{if } d = 2 \mod 3. \end{cases} \tag{4.46}$$

Analogous identities follow from (4.42) for $s=q-2$, $s=q-3$ and $s=q-4$ (assuming that $s \in I_s$), where the values of $c^{(d)}(2)$, $c^{(d)}(3)$, and $c^{(d)}(4)$ were given in (2.19)–(2.21) of Ref. [19]. Thus, in using the results above for $n_{Zh}(L_y, d, s)$, $n_{Ph}(L_y, d, s)$, $N_{Zh,L_y,s}$, and $N_{Ph,L_y,s}$, it is understood that they apply for generic values of $s \in I_s$ but involve simplifications for special values of s. We shall give an example of this in the next section.

5 Circuit Graphs C_n

The circuit graph C_n, or equivalently, the 1D lattice with periodic boundary conditions, has chromatic number

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \geq 2 \text{ is even} \\ 3 & \text{if } n \geq 3 \text{ is odd.} \end{cases} \tag{5.1}$$

(The case $n = 1$ is a single vertex with a loop, for which there is no proper q-coloring, so $Ph(C_1, q, w)$ vanishes identically.) In general, $Z(C_n, q, s, v, w)$ has the structure

$$Z(C_n, q, s, v, w) = \sum_{j=1}^{s+1} [\lambda_{Z,1,0,j}(q, s, v, w)]^n + \tilde{c}^{(1)} v^n \tag{5.2}$$

where we recall that $\tilde{c}^{(1)} = q - s - 1$. We find that (suppressing arguments) the $\lambda_{Z,1,0,j}$ for $1 \leq j \leq s + 1$ are given by

$$\begin{aligned} \lambda_{Z,1,0,j} = \frac{1}{2}\big[&q - s + v + w(s + v) \\ &\pm [\{q - s + v + w(s + v)\}^2 - 4vw(q + v)]^{1/2}\big] \end{aligned} \tag{5.3}$$

for $j = 1, 2$ (which is the total set for $s = 0$ or $s = 1$), and, for $s \geq 2$,

$$\lambda_{Z,1,0,j} = vw \quad \text{for } 3 \leq j \leq s + 1. \tag{5.4}$$

That is, for general $s \in I_s$,

$$Z(C_n, q, s, v, w) = \sum_{j=1}^{2} [\lambda_{Z,1,0,j}]^n + (s - 1)(vw)^n + (q - s - 1)v^n. \tag{5.5}$$

It is readily checked that this expression for $Z(C_n, q, s, v, w)$ (i) reduces to the zero-field Potts model partition function

$$Z(C_n, q, v) = (q + v)^n + (q - 1)v^n \tag{5.6}$$

for $s = 0$ or $w = 1$, (ii) satisfies the general symmetry property (2.13), and (iii) reduces to $w^n Z(C_n, q, v)$ for $s = q$, in agreement with (2.18).

As was noted briefly at the beginning of the paper, our result (5.5) shows a qualitative difference between the case $s = 1$ considered previously [2] and the more general set of cases with $s \geq 2$ in the interval I_s, namely the fact that the third term, $(s - 1)(vw)^n$, is absent for $s = 1$ but is present for other values of $s \in I_s$. By the $s \leftrightarrow q - s$ symmetry inherent in (2.4) and (2.13), this also means that another term vanishes identically for $s = q - 1$ but is present for other values of $s \in I_s$; this is the last term in (5.5), $(q - s - 1)v^n$. It is interesting to observe that the symmetry (2.13) applies not just to the total $Z(C_n, q, s, v, w)$, but also to parts of this function. Specifically, under the replacement $s \to q - s$, one sees that (i) the sum of the last two terms in (5.5), $(s-1)(vw)^n + (q-s-1)v^n$, transforms into $w^n[(q-s-1)v^n + (s-1)(vw^{-1})^n]$ and (ii) the first two terms, $\sum_{j=1}^{2}[\lambda_{Z,1,0,j}(q,s,v,w)]^n$ transform into $w^n \sum_{j=1}^{2}[\lambda_{Z,1,0,j}(q,q-s,v,w^{-1})]^n$, so that each of these parts, (i) and (ii), individually satisfies the symmetry (2.13). For the illustrative case $v=-1$, $s=1$, and $w=3/2$, the magnitudes $a=|\lambda_{Z,1,0,1}|$, $b=|\lambda_{Z,1,0,2}|$, and $c=1$ were plotted in Fig. 5 of Ref. [2]. In the figure, $b>a>c$ for $1.8<q<2$; $b>c>a$ for $1<q<1.8$; $c>b>a$ for $\sqrt{3}-1<q<1$; and $a=b<c$ for $1/3<q<\sqrt{3}-1$ (which corrects some misprints in the caption).

We exhibit $Z(C_n,q,s,v,w)$ for $2 \le n \le 4$ below. To keep the equations as compact as possible, we write the coefficients of the terms of maximal degree in w and of degree 0 in w in terms of zero-field partition functions using the general results (2.35) and (2.31). We find

$$\begin{aligned} Z(C_2,q,s,v,w) &= Z(C_2,s,v)w^2 + 2s(q-s)w + Z(C_2,q-s,v) \\ &= q^2 + [2t + v(v+2)]q + t[t + v(v+2)(w+1)], \end{aligned} \tag{5.7}$$

$$\begin{aligned} Z(C_3,q,s,v,w) &= Z(C_3,s,v)w^3 + 3s(q-s)(s+v)w^2 \\ &\quad + 3s(q-s)(q-s+v)w + Z(C_3,q-s,v), \end{aligned} \tag{5.8}$$

$$\begin{aligned} Z(C_4,q,s,v,w) &= Z(C_4,s,v)w^4 + 4s(q-s)(s+v)^2w^3 \\ &\quad + 2s(q-s)[3(q^2-s^2-(q-s)^2) + 4qv + 4v^2]w^2 \\ &\quad + 4q(q-s)(q-s+v)^2w + Z(C_4,q-s,v). \end{aligned} \tag{5.9}$$

As usual, one obtains the $Ph(C_n,q,s,w)$ for each n by setting $v=-1$ in $Z(C_n,q,s,v,w)$. For $s=1$, the parts of $Z(C_n,q,1,v,w)$ were given in Ref. [3] and $Ph(C_n,q,1,w)$ was given in Ref. [2]. Magnitudes of λ's for C_n with $s=1$ were plotted in Fig. 5 of Ref. [2]. (As is evident from that figure, in the notation.)

In the context of weighted-set coloring, so that s is an integer in the interval I_s, it was noted [2] that for $s=1$, $Ph(C_n,q,1,w)$ contains the factor $(q-1)$. It was also noted that if n is odd, say $n=2m+1$ with $m=1,2,\ldots$, then $Ph(C_{2m+1},q,1,w)$ also contains the factor $(q-2)$, so that for odd n, $Ph(C_{2m+1},2,1,w)=0$ [2]. The fact that if $n=2m+1$ is odd and $q=2$, then one cannot perform a proper q-coloring of C_n, so that $Ph(C_{2m+1},2,s,w)=0$, is independent of both w and s. However, in contrast to the $s=1$ case, where the vanishing of $Ph(C_{2m+1},q,1,w)$ for $q=2$ occurred via a factor of $(q-2)$, this is not the case for general s. Instead, $Ph(C_{2m+1},q,s,w)$ is such that if one evaluates it at $q=2$, there is a factor which is a polynomial in s that implicitly but necessarily vanishes. This vanishing occurs because of the implicit restriction on the values of s, namely that s is an integer in the interval $0 \le s \le q$. For example, consider $Ph(C_3,q,s,w)$, given above as the $v=-1$ special case of (5.8). Evaluating this at $q=2$, we obtain

$$Ph(C_3,2,s,w) = s(s-1)(s-2)(w-1)^3. \tag{5.10}$$

This implicitly vanishes for any of the allowed (integer) values of s in the set I_s because for $q=2$, s can only take on the values 0, 1, or 2 in this set. The same type of mechanism is responsible for the vanishing of $Ph(C_{2m+1},q,s,w)$ at $q=2$ for higher values of m. For $s=2$, if $n \ge 3$ is odd, then our results show that $Ph(C_n,q,s,w)$ contains the factor $(q-2)$. If $s \ge 3$, then, in general, $Ph(C_n,q,s,w)$ does not have such overall factors. The reason for this is that s is bounded above by q, so that if $s \ge 3$, then $q \ge 3$. This is equal to $\chi(C_n)$ for

n odd and greater than $\chi(C_n)$ for n even (cf. (5.1)), so that $Ph(C_n, q, s, w)$ is not forced to vanish the way it is for the cases of $s = 0,\ 1, 2$ when q can be less then $\chi(C_n)$.

We have mentioned above that our general structural formulas for $Z(G, q, s, v, w)$ and $Ph(G, q, s, w)$ with G a cyclic strip graph simplify considerably when $s = q$, as required by the general relations (2.18) and (1.5). It is interesting to see how this occurs for this family of circuit graphs, $G = C_n$. There are $N_{Zh,L_y,s} = s + 2$ terms $\lambda_{Z,L_y,d,j}(q, s, v, w)$ with $L_y = 1$ whose n'th powers occur in (5.2). Of these, $n_{Zh,1,0,s} = s + 1$ multiply the coefficient $\tilde{c}^{(0)} = 1$ and the remaining term, $[\lambda_{Z,1,1}(q, s, v, w)]^n = v^n$, multiplies the coefficient $\tilde{c}^{(1)} = q - s - 1$. From the general relation (2.18), we know that for $s = q$, $Z(C_n, q, q, v, w) = w^n Z(C_n, q, v)$, where $Z(C_n, q, v)$ is the zero-field Potts model partition function for the circuit graph C_n, given by (5.6). Hence, we can deduce that of the $s + 1$ terms $\lambda_{Z,1,0,j}(q, s, v, w)$, (i) one becomes equal to $w(q + v)$; (ii) a second becomes equal to v, and (iii) the remaining $s - 1 = q - 1$ terms become equal to wv. Using the fact that $s = q$, so that $\tilde{c}^{(1)} = -1$, we then have the reduction

$$
\begin{aligned}
Z(C_n, q, q, v, w) &= \sum_{j=1}^{s+1} [\lambda_{Z,1,0,j}(q, q, v, w)]^n + \tilde{c}^{(1)} v^n \\
&= [w(q + v)]^n + v^n + (q - 1)(wv)^n - v^n \\
&= w^n [(q + v)^n + (q - 1)v^n] \\
&= w^n Z(C_n, q, v). \qquad (5.11)
\end{aligned}
$$

Thus, the identity (2.18) (a special case of the symmetry (2.13)) is realized via a "transmigration" process in which one or more $\lambda_{Z,L_y,d,j}(q, s, v, w)$'s for a given d become equal or proportional to (respectively, one or more) $\lambda_{Z,L_y,d',j}(q, s, v, w)$'s for a different d' and hence their m'th powers can be regrouped with the latter, thereby changing the effective coefficients that multiply the $[\lambda_{Z,L_y,d',j}(q, s, v, w)]^m$'s , where here $m = n$. This transmigration process is a general one and occurs also for higher values of strip width L_y. This process is also the mechanism whereby the identities (2.13) and (2.14) are satisfied. Thus, the results above for $n_{Zh}(L_y, d, s)$, $n_{Ph}(L_y, d, s)$, $N_{Zh,L_y,s}$, and $N_{Ph,L_y,s}$ apply for generic values of s, but there are simplifications, involving this type of transmigration, for special values of s.

6 Some Properties of the Zeros of $Ph(G, q, s, w)$

One can solve for the zeros of $Z(G, q, s, v, w)$ as functions of any of the four variables q, s, v and w with the other three held fixed, and similarly, one can solve for the zeros of $Ph(G, q, s, w)$ as a function of any of the three variables q, s, and w with the other two held fixed. Here it is understood that one uses (2.6), with these four variables each generalized to lie in $\mathbb{C}$.

In general, some zeros in w and s can have unbounded magnitudes as a function of the other variables. The underlying reason for this can be seen at an algebraic level as the fact that the coefficient of the highest-degree term in this variable in $Z(G, q, s, v, w)$ can vanish at some values of w and/or s. Related to this are the facts that (i) at the special values $w = 1$ and $w = 0$, $Z(G, q, s, v, w)$ loses its dependence on s, and (ii) at the special value $s = 0$, $Z(G, q, s, v, w)$ loses its dependence on w. As a consequence, the zeros in these variables move off to infinity. Similar statements apply for $Ph(G, q, s, w)$. We proceed to discuss some properties of these zeros.

6.1 Zeros of $Z(G,q,s,v,w)$ and $Ph(G,q,s,w)$ in q

Here we consider the zeros of $Z(G,q,s,v,w)$ and $Ph(G,q,w)$ in q, as a function of s and w, for some graphs G. Since the maximal degree of $Z(G,q,s,v,w)$ and $Ph(G,q,s,w)$ in the variable q is $n(G)$, each of these polynomials has this number of zeros in the variable q. In general, since $Z(G,q,s,v,1)=Z(G,q,v)$ while $Z(G,q,s,v,0)=Z(G,q-s,v)$, it follows that as w decreases from 1 to 0, there is an overall shift to the right in the zeros of $Z(G,q,s,v,w)$ in the q plane by s units, and this holds, in particular, for the case $v=-1$ that yields $Ph(G,q,s,w)$. This shift is illustrated by some simple examples. For $G=L_1=K_1=N_1$, $Z(L_1,q,s,v,w)=0$ at

$$q_{L1z}=-t=s(1-w). \tag{6.1}$$

This increases from 0 to s as w decreases from 1 to 0 in the DFSCP interval and decreases from 0 through negative values as w increases above 1 in the FSCP interval. For $L_2=K_2$, $Z(L_2,q,s,v,w)$ vanishes at

$$q_{L2z,j}=\frac{1}{2}\left[-v+2s(1-w)\pm\sqrt{v[v-4sw(w-1)]}\right] \tag{6.2}$$

where $j=1,2$ for the $\pm$ sign. As w decreases from 1 to 0, the root $q_{L2z,1}$ increases from 0 to s, while the root $q_{L2z,2}$ increases from $-v$ to $s-v$. The zeros of graph-coloring polynomials in q satisfy certain boundedness properties for fixed nonzero magnetic field [21]. However, the zeros of $Z(G,q,s,v,w)$ and $Ph(G,q,s,w)$ in q are, in general, unbounded as $|w|\to\infty$. This is already evident in the simplest case of a single vertex, for which the zero of $Z(L_1,q,s,v,w)$ at $q=s(1-w)$ has a magnitude that, for $s\neq 0$, goes to infinity as $|w|\to\infty$.

6.2 Zeros of $Z(G,q,s,v,w)$ and $Ph(G,q,s,w)$ in s

One may also study the zeros of $Z(G,q,s,v,w)$ in each of the three variables s, v, and w with the other two (and q) held fixed, and the zeros of $Ph(G,q,s,w)$ in each of the two variables s and w with the other (and q) held fixed. In general, these zeros are unbounded in magnitude even when the other variables vary over a finite range. The reason for the divergences in these zeros of $Z(G,q,s,v,w)$ is the fact that the coefficient of the term in $Z(G,q,s,v,w)$ of highest degree in the given variable can vanish as one changes other variables, and similarly with $Ph(G,q,s,w)$. Again, this can be illustrated with simple examples.

We begin our discussion with the zeros in s, where it is understood that this variable is formally extended from the integers in the interval I_s to the complex numbers. The Potts model partition function for the single-vertex graph L_1, $Z(L_1,q,s,v,w)$, vanishes at

$$s=\frac{q}{1-w}. \tag{6.3}$$

For $q\neq 0$, this diverges as $w\to 1$. For L_2, $Z(L_2,q,s,v,w)$ has zeros in s at

$$s_{L2z,j}=\frac{-[2q+v(w+1)]\pm\sqrt{v[v(w+1)^2+4qw]}}{2(w-1)} \tag{6.4}$$

where $j = 1, 2$ correspond to the $\pm$ signs, respectively. We find that

$$s_{L2z,j} \sim \frac{-(q+v) \pm \sqrt{v(q+v)}}{w-1} \quad \text{as } w \to 1 \tag{6.5}$$

so that as $w \to 1$, the magnitudes $|s_{L2z,j}|$ are, in general, unbounded. These results for the full Potts model partition function apply, *a fortiori*, to the special case $v = -1$ that defines the weighted-set chromatic polynomial, $Ph(L_2, q, s, w)$. Above, we have explained the origin of this type of divergence as being due to the fact that the coefficient of the highest-power term of $Z(G, q, s, v, w)$ and $Ph(G, q, s, w)$ in the given variable, here s, vanishes. This occurs at $w = 1$. Closely related to this, at $w = 1$, all dependence on s in $Z(G, q, s, v, w)$ disappears, so it is understandable that the zeros in s would disappear by moving off to infinity in this limit.

6.3 Zeros of $Z(G, q, s, v, w)$ and $Ph(G, q, s, w)$ in w

Here we comment on zeros of $Z(G, q, s, v, w)$ and $Ph(G, q, s, w)$ in w. First, we note that the symmetry (2.13) implies that if one replaces s by $q - s$, then the zeros of $Z(G, q, s, v, w)$ and $Ph(G, q, s, w)$ in w away from the origin map into their inverses. In particular, if $q = 2s$, then (2.13) reads $Z(G, 2s, s, v, w) = w^n Z(G, 2s, s, v, w^{-1})$, so that the zeros of Z away from the origin in the w plane form a set that is invariant under inversion. For $G = L_1$, $Z(L_1, q, s, v, w)$ vanishes at

$$w_{L1z} = 1 - \frac{q}{s}. \tag{6.6}$$

From (3.2), we find that $Z(L_2, q, s, v, w) = 0$ for

$$w_{L2z,j} = \frac{s(s-q) \pm \sqrt{s(s-q)v(q+v)}}{s(s+v)} \tag{6.7}$$

where $j = 1, 2$ correspond to the $\pm$ signs, respectively. The right-hand sides of (6.6) and (6.7) both diverge as $s \to 0$. In particular,

$$w_{L2z,j} \sim \pm\sqrt{\frac{(-q)(q+v)}{sv}} \quad \text{as } s \to 0 \tag{6.8}$$

for $j = 1, 2$. The fact that, in general, the magnitudes of the zero w_{L1z} of $Z(L_1, q, s, v, w)$ and the zeros $w_{L2z,j}$, $j = 1, 2$, of $Z(L_2, q, s, v, w)$ diverge as $s \to 0$ is again understandable, since for a graph G, if $s = 0$, then $Z(G, q, s, v, w)$ reduces to $Z(G, q, v)$, with no dependence on w. Hence, in the limit $s \to 0$, it is natural that the zeros in w disappear by moving off to infinity. Finally, as $s \to -v$, we find that

$$w_{L2z,2} \to \frac{q+2v}{2v} \quad \text{as } s \to -v \tag{6.9}$$

while $w_{L2z,1}$ is unbounded:

$$w_{L2z,1} \sim -\frac{2(q+v)}{s+v} \quad \text{as } s \to -v. \tag{6.10}$$

6.4 Zeros of $Z(G,q,s,v,w)$ in v

To illustrate the calculation of the zeros of $Z(G,q,s,v,w)$ in v, we again use our simple example graph, L_2. We find the zero of $Z(L_2,q,s,v,w)$ in v occurs at

$$v = -\frac{[q+s(w-1)]^2}{q+s(w-1)(w+1)}. \tag{6.11}$$

This has an unbounded magnitude if $q+s(w^2-1)=0$, i.e.,

$$s = \frac{q}{1-w^2}. \tag{6.12}$$

This divergence in the magnitude of the right-hand side of (6.11) does not directly affect the weighted-set proper vertex coloring of this L_2 graph because in the DFSCP region $0 \le w < 1$, the condition (6.12) implies that $s > q$, outside the actual coloring interval I_s, and in the FSCP region $w > 1$, it implies that s is negative, again outside of this interval I_s.

7 Quantities Defined in the Limit $n(G) \to \infty$

7.1 f and Φ Functions

Let us consider families of graphs G_m that can be built up recursively, such as lattice strips. For such graphs, the $m+1$ member of the family is obtained from the m th member by (possibly cutting and) gluing in a given subgraph. For example, for the graph C_m, one cuts the circuit at some vertex and inserts another edge and vertex to get C_{m+1}, and so forth. Generalizing this, another example is a strip of the square lattice with transverse width L_y, length $L_x = m$, and periodic longitudinal boundary conditions, which we denote as $sq(L_y \times L_x, cyc)$. In this case, the number of vertices is $n = L_y L_x$. Generically, the number of vertices of a recursive graph G_m is of the form $n(G_m) = am + b$, where a and b are (integer) constants, so that the limit $n \to \infty$ is equivalent to the limit $m \to \infty$. We denote the formal $n \to \infty$ limit of such graphs as $\{G\} = \lim_{m\to\infty} G_m$. In the present context, this $n \to \infty$ limit corresponds to the limit of infinite length for a strip graph of fixed width and some prescribed boundary conditions. Correspondingly, one can define the free energy per vertex as follows (with the subscript m suppressed in the notation)

$$f(\{G\},q,s,v,w) = \lim_{n\to\infty} n^{-1} \ln[Z(G,q,s,v,w)] \tag{7.1}$$

and a function $\Phi(\{G\},q,s,w)$,

$$\Phi(\{G\},q,s,w) = \lim_{n\to\infty} [Ph(G,q,s,w)]^{1/n}. \tag{7.2}$$

As before (cf. (1.9) of [22] and (2.8) of [23] and [24–26]), one must take account of a noncommutativity of limits that can occur, namely the fact that for certain special values of q, denoted $\{q_{sp}\}$, the limits $n \to \infty$ and $q \to q_{sp}$ do not commute:

$$\lim_{n\to\infty} \lim_{q\to q_{sp}} Z(G,q,s,v,w)^{1/n} \neq \lim_{q\to q_{sp}} \lim_{n\to\infty} Z(G,q,s,v,w)^{1/n} \tag{7.3}$$

and the analogous formulas for the $v=-1$ case which defines $\Phi(\{G\},q,s,w)$. For further details, we refer the reader to our previous discussions of this [2, 22, 23]. An explicit example is provided by our result for $Z(C_n,q,s,v,w)$ in (5.5); if one sets $q=s+1$ first before taking $n\to\infty$, then the last term drops out, while if one takes $n\to\infty$ first with $q\neq s+1$, then, since $\lim_{n\to\infty}[(q-s-1)v^n]^{1/n}=v$, the last term may remain in f. We see also an additional type of noncommutativity that is present, namely that if we extend s from an integer in I_s to a real (or complex) variable, then for a set of special values of s, denoted $\{s_{sp}\}$,

$$\lim_{n\to\infty}\lim_{s\to s_{sp}} Z(G,q,s,v,w)^{1/n}\neq \lim_{s\to s_{sp}}\lim_{n\to\infty} Z(G,q,s,v,w)^{1/n}. \tag{7.4}$$

Again, our result for $Z(C_n,q,s,v,w)$ in (5.5) provides an illustration of this; if one sets $s=1$ first before taking $n\to\infty$, then the third term, $(s-1)(vw)^n$, drops out, while if one takes $n\to\infty$ first with $s\neq 1$, and then sets $s=1$, it follows that, since $\lim_{n\to\infty}[(s-1)(vw)^n]^{1/n}=vw$, the resultant limiting term vw may remain in f. Similarly, if one sets $s=q-1$ before taking $n\to\infty$, then the last term, $(q-s-1)v^n$ drops out, while if one takes $n\to\infty$ first with $s\neq q-1$, and then sets $s=q-1$, it follows that since $\lim_{n\to\infty}[(q-s-1)v^n]^{1/n}=v$, the resultant limiting term v may remain in f. In our analysis of the $n\to\infty$ limit for recursive families of graphs, unless otherwise indicated, we shall choose the order of limits in which we fix s first and then take $n\to\infty$.

The function $Ph(\{G\},q,s,w)$ generalizes the ground state degeneracy of the zero-field, zero-temperature Potts antiferromagnet, $W(\{G\},q)=\lim_{n\to\infty}P(G,q)^{1/n}$. Thus (with care taken concerning the above-mentioned noncommutativity of limits), $\Phi(\{G\},q,s,1)=\Phi(\{G\},q,0,w)=W(\{G\},q)$. In the case of the zero-field, zero-temperature Potts antiferromagnet, the associated configurational entropy per vertex (which is thus the ground-state entropy per site) $\{G\}$ is $S=k_B\ln W$. The third law of thermodynamics states that the entropy per site S should vanish as the temperature goes to zero. However, there are a number of exceptions to this law. For instance, for the zero-field q-state Potts antiferromagnet on a square lattice, an elementary argument yields the lower bound $S/k_B\geq(1/2)\ln(q-1)$, which is nonzero for $q\geq 3$.

In the present case of the weighted-set chromatic polynomial, let us consider first the FSCP interval $w>1$ and assume that $q\geq 2$, so that a proper q-coloring can be performed for a bipartite graph. We also assume that s is an integer for this discussion and lies in the interval $2\leq s\leq q$. Applying our lower bound (2.78) to a particular bipartite graph, namely the strip graph of the square (*sq*) lattice with width L_y vertices and even length L_x vertices, denoted $sq(L_y\times L_x)$, we have (with $n_1=n_2=L_xL_y/2$) the lower bound

$$Ph(sq(L_y\times L_x),q,s,w)\geq s(s-1)^{n/2}w^n. \tag{7.5}$$

Taking $L_x\to\infty$ with L_y fixed, we thus obtain the lower bound $\Phi(sq(L_y\times\infty),q,s,w)\geq w\sqrt{s-1}$. Hence, in this limit, the entropy is bounded below by

$$S(sq(L_y\times\infty),q,s,w)\geq \ln w+\frac{1}{2}\ln(s-1)\quad\text{(FSCP case)}. \tag{7.6}$$

For $s\geq 3$ (which implies $q\geq 3$ also), this ground state entropy is nonzero. In the DFSCP interval $0\leq w<1$, with $1\leq s\leq q-3$, applying our lower bound (2.79) to the same limit of this strip graph, we have $\Phi(sq(L_y\times\infty),q,s,w)\geq\sqrt{q-s-1}$, so that

$$S(sq(L_y\times\infty),q,s,w)\geq \frac{1}{2}\ln(q-s-1)\quad\text{(DFSCP case)}. \tag{7.7}$$

For the given range, $q\geq s+3$, this entropy is again nonzero.

7.2 Example of Calculation of f and Φ for a Family of Graphs

We illustrate the calculation of the functions f and Φ for the $n \to \infty$ limit of the circuit graph C_n, or equivalently, the one-dimensional lattice with periodic boundary conditions. We shall take s to be a fixed integer in the interval $0 \le s \le q$ for this analysis. For a given set of values of q, s, v, and w, the functional form of f is determined by the term $\lambda_{Z,1,0,j}(q,s,v,w)$ in (5.5) with the largest magnitude. For fixed s, v, and w and sufficiently large real q, this is $\lambda_{Z,1,0,1}$. Following our nomenclature in earlier work for $w=1$, we denote this region as region R_1. As for the zero-field case, f and Φ in this region are the same for the $n \to \infty$ limit of the line graph L_n and the circuit graph C_n. We denote these limits as $\{L\}$ and $\{C\}$. We thus have

$$f(\{L\},q,s,v,w) = f(\{C\},q,s,v,w) = \ln[\lambda_{Z,1,0,1}(q,s,v,w)] \tag{7.8}$$

and

$$\Phi(\{L\},q,s,w) = \Phi(\{C\},q,s,w) = \ln[\lambda_{Ph,1,0,1}(q,s,w)] \tag{7.9}$$

where $\lambda_{Z,1,0,j}(q,s,v,w)$, $j=1,2$, were given for this family of graphs in (5.3) and $\lambda_{Ph,1,0,j}(q,s,w) = \lambda_{Z,1,0,j}(q,s,-1,w)$.

It is of interest to consider how $\Phi(G,q,s,w)$ for $\{G\}=\{L\}$ (or equivalently $\{G\}=\{C\}$ in region R_1) behaves for certain special cases or limits of its variables. For example, in the limit where the weighting is removed, i.e., for $w \to 1$, $\Phi(\{L\},q,s,w)$ has the Taylor series expansion

$$\Phi(\{L\},q,s,w) = q-1+\frac{s(q-1)(w-1)}{q} - \frac{s(q-1)(q-s)(w-1)^2}{q^3} + O\big((w-1)^3\big) \quad \text{as } w \to 1. \tag{7.10}$$

For $s=1$, it was shown in Ref. [2] that the asymptotic behavior of $\Phi(\{L\},q,s,w)$ for large $|w|$ is

$$\Phi(\{L\},q,s,w) \sim \sqrt{(q-1)w}\left[1+O\left(\frac{1}{\sqrt{w}}\right)\right] \quad \text{for } s=1 \text{ and } |w|\to\infty. \tag{7.11}$$

In contrast, for $s \ne 1$, we find

$$\Phi(\{L\},q,s,w) \sim (s-1)w + \frac{q-s-1}{2} + \left[\frac{s(q-s)+q-1}{2(s-1)}\right] + O\left(\frac{1}{w}\right). \tag{7.12}$$

As is evident, the large-$|w|$ behavior is different, depending on whether or not $s=1$. For $|q| \to \infty$, we obtain the asymptotic expansion

$$\Phi(\{L\},q,s,w) \sim q + s(w-1) - 1 - \frac{sw(w-1)}{q} + O\left(\frac{1}{q^2}\right). \tag{7.13}$$

Extending s from an integer in the interval I_s to a real (or complex) number, it is of interest to determine the limiting behavior of $\Phi(\{L\},q,s,w)$ as $s \to 0$ and $s \to q$. We calculate

$$\Phi(\{L\},q,s,w) = q-1+\frac{(w-1)(q-1)s}{w+q-1} + O(s^2) \quad \text{as } s \to 0 \tag{7.14}$$

and

$$\Phi(\{L\}, q, s, w) = w(q-1) - \frac{w(w-1)(q-1)(q-s)}{w(q-1)+1} + O\left((q-s)^2\right) \quad \text{as } s \to q. \tag{7.15}$$

From our study of $\Phi(\{G\}, q, s, w)$ for the $n \to \infty$ limits of various families of graphs, we have observed several generic properties. Let us consider families of strip graphs G of regular lattices Λ. Let the chromatic number of the lattice Λ be denoted as $\chi(\Lambda)$ and assume that $q \ge \chi(\Lambda)$. A technical assumption is that the $n \to \infty$ limit of the lattice strip graphs is taken in a manner such that for each G, $\chi(G) = \chi(\Lambda)$. (For example, for square-lattice strip graphs with periodic longitudinal boundary, this means taking the length to be even.) Then for fixed $s \in I_s$ we have observed that (i) for fixed $w > 0$, $\Phi(\{G\}, q, s, w)$ is a monotonically increasing function of q and (ii) for fixed q, $\Phi(\{G\}, q, s, w)$ is a monotonically increasing function of w for $w > 0$. One can also analyze the behavior of $\Phi(\{G\}, q, s, w)$ as a function of s for fixed q and w, but because of the noncommutativity (7.4), one must take care in specifying the order of limits used in defining this function. The order that we take is first to set s to a given value in I_s and then to take $n \to \infty$. As is evident in the definition of I_s, it is understood that $s \le q$. We find that if $w > 1$, then $\Phi(\{G\}, q, s, w)$ is an increasing function of $s \in I_s$, while if $0 \le w < 1$, then $\Phi(\{G\}, q, s, w)$ is a decreasing function of $s \in I_s$.

Focusing on the circuit graph, one sees that for values of the variables such that another term λ in (5.5) becomes dominant, there is a non-analytic change in f and Φ. As we have discussed earlier, this is also associated with a locus, denoted generically $\mathcal{B}$, that comprises the accumulation set of zeros of the respective function, f and Φ, For definiteness, we analyze the locus $\mathcal{B}$ in the q plane (denoted $\mathcal{B}_q$) for $\Phi(\{C\}, q, s, w)$, with s and w fixed. It may be recalled that for the $n \to \infty$ limit of the unweighted chromatic polynomial, $\mathcal{B}_q$ is the unit circle $|q-1| = 1$, so $q_c = 2$ in that case [22, 27]. For the weighted-set chromatic polynomial, $Ph(C_n, q, s, w)$, the accumulation locus $\mathcal{B}$ depends on the value of s and on whether w is in the DFSCP interval $0 \le w < 1$ or the FSCP interval $w > 1$. We consider here the DFSCP interval, since as w decreases from 1 to 0, $Ph(C_n, q, s, w)$ interpolates between $P(G, q)$ and $P(G, q-s)$. We also generally take $s \ne 0$, since for $s = 0$ $Ph(C_n, q, s, w)$ reduces to the well-studied unweighted chromatic polynomial $P(C_n, q)$. (However, our results subsume this $s = 0$ case.)

In this DFSCP interval, for $s = 1$ or $s = 2$, the lower boundary of the region R_1 on the real axis, denoted q_c, is determined by the equality in magnitude

$$|\lambda_{Ph,1,0,1}| = |\lambda_{Ph,1,1}| = 1, \tag{7.16}$$

which yields the result

$$q_c = 2 + \frac{s(1-w)}{1+w} \quad \text{for } \{G\} = \{C\} \text{ and } s = 1 \text{ or } s = 2 \text{ and } 0 \le w \le 1. \tag{7.17}$$

For $0 \le w < 1$, this value of q_c is greater than the value $q_c = 2$ for the unweighted chromatic polynomial. Furthermore, as is evident from (7.17), q_c is a monotonically increasing function of s for fixed w in this DFSCP interval and a monotonically decreasing function of w for the fixed values of s given above. As w decreases from 1 to 0, q_c increases continuously from 2 to $2+s$. In contrast, the left-hand part of the boundary changes discontinuously; as w decreases by an arbitrarily small amount below 1, the point on the left where $\mathcal{B}_q$ crosses the real q axis jumps discontinuously from $q = 0$ to $q = s$. These results are in accord with the fact that for $w=0$, $\mathcal{B}_q$ is the locus of solutions to the equation $|q-(1+s)|=1$, i.e., the unit circle in the q plane centered at the point $q=1+s$, crossing the real axis on the left at $q=s$ and on the right at $q=2+s$. The locus $\mathcal{B}_q$ separates the q plane into two regions. We label the regions outside and inside the closed curve $\mathcal{B}$ as R_1 (noted before) and R_2, respectively. For $s=1$, there are several noncommutativity effects relevant for $\mathcal{B}_w$ [2]. If, for example, one takes $n\to\infty$ first and then sets $q=2$, the relevant terms are $|\lambda_{Ph,1,0,j}|=|\sqrt{w}|$ and $|\lambda_{Ph,1,1}|=1$, so that $\mathcal{B}_w$ is the unit circle $|w|=1$.

For $s>2$, there is a change in the locus $\mathcal{B}_q$, because the condition of degeneracy of leading λ's is different; rather than (7.16), it takes the form

$$|\lambda_{Ph,1,0,1}|=|\lambda_{Ph,1,0,2}|. \tag{7.18}$$

This entails the condition that (i)

$$q-s-1+w(s-1)=0, \tag{7.19}$$

so that $\lambda_{Ph,1,0,1}=-\lambda_{Ph,1,0,2}$, and the condition that (ii) $|\lambda_{Ph,1,0,1}|=|\lambda_{Ph,1,0,2}|>1$, so that these λ's are dominant. Substituting for q from (7.19), we find that condition (ii) is satisfied in the relevant range of w for

$$\frac{1}{s-1}<w<1. \tag{7.20}$$

This interval is nonvanishing if $s>2$. Thus, for $s>2$, provided that conditions (i) and (ii) are satisfied, $\mathcal{B}_q$ has the form of an open self-conjugate arc that crosses the real axis at the point given by (7.19), so that in this case,

$$q_c=s+1-w(s-1) \quad \text{for } \{G\}=\{C\} \text{ and } s>2 \ \text{ and } \frac{1}{s-1}<w<1. \tag{7.21}$$

This self-conjugate arc is concave to the left and ends at the arc endpoints given by where the ($v=-1$ evaluation of the) expression in the square root of (5.3) vanishes, namely

$$q_e, q_e^*=(s+1)(1-w)\pm 2i\sqrt{sw(1-w)}. \tag{7.22}$$

This locus does not separate the q plane into different regions. It is straightforward to carry out a similar analysis of $\mathcal{B}_q$ for the FSCP regime $w>1$.

8 Related Topics

As our results show, one finds a number of intriguing features in the study of weighted-set vertex coloring of graphs. There are many further directions of research in this general area. One could, for example, use the methods presented here to calculate $Z(G,q,s,v,w)$ and $Ph(G,q,s,w)$ for other individual graphs and families of graphs. One could also investigate further the zeros of these functions in various variables and their accumulation sets $\mathcal{B}$ for recursive graphs in the limit of infinitely many vertices. It would, moreover, be worthwhile to study connections with weighted loop models [28, 29]. One could also investigate a different but related type of graph coloring problem in which the set of colors that one chooses from to assign to each vertex depends on the vertex. The unweighted case is called the list coloring problem in graph theory [30], and it would be useful to study the weighted-set generalization of list coloring. We are pursuing these studies.

9 Conclusions

In this paper we have studied the weighted-set graph coloring problems, in which one assigns q colors to the vertices of a graph such that adjacent vertices have different colors, with a vertex weighting w that either disfavors or favors a given set of s colors. In particular, we have analyzed an associated weighted-set chromatic polynomial $Ph(G, q, s, w)$ and have also related this to a corresponding Potts model partition function with external magnetic fields, $Z(G, q, s, v, w)$. These functions exhibit a wealth of interesting properties. We have proved various general results on these and illustrated them for particular graphs and families of graphs.

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